Interest rates and the existence of stationary equilibria in incomplete insurance-market economies

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Abstract

We prove that the difference between the interest rate and the discount rate is proportional to the severity of credit constraints at any stationary equilibrium in standard heterogeneous-agent economies. This severity is measured as the economy-wide average of the shadow price of the credit constraint. We deduce that stationary equilibria only exist when credit constraints are binding for a positive mass of agents. This has important implications for both positive and normative results in heterogeneous-agent models.

Keywords: Incomplete markets, interest rate, existence.

JEL codes: E21, E44, D91, D31.

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1 Introduction

Incomplete insurance-market economies are fast becoming a standard setup for macroeconomic analysis. Such frameworks are also called Bewley-Huggett-Aiyagari-Imohoroğlu economies (named after the seminal papers of Bewley (1983), Huggett (1993), Imrohoroğlu (1992), and Aiyagari (1994)) or, more concisely, heterogeneous-agent economies. These economies have the great advantage of being able to reconcile a sound theoretical model with the actual heterogeneity observed in micro-data. However, these frameworks are highly complex and some theoretical aspects – including equilibrium existence and multiplicity – are not yet fully understood. This article provides a novel characterization result stating that, at any stationary equilibrium, the difference between the discount rate $1/\beta$ and the interest rate $1+r$ is proportional to the severity of average credit constraints in the economy. This severity is measured by the average shadow price (i.e., Lagrange multiplier) of the household borrowing constraint, where the average is computed over the whole population.

This characterization has several implications. First, the severity of the credit constraints and the average marginal utility of households are sufficient to pin down the liquidity premium $\frac{1}{\beta} - (1 + r)$. Second, if binding credit constraints are not experienced by a positive mass of households, the liquidity premium is null and the stationary equilibrium is characterized by $\beta(1 + r) = 1$. As shown in Chamberlain and Wilson (2000), this implies that the stationary equilibrium does not exist in standard cases. Stationary equilibria in heterogeneous-agent models therefore go hand-in-hand with binding credit constraints.

The setup of our proof comprises a standard heterogeneous-agent economy, where agents, facing a Markovian idiosyncratic productivity risk and borrowing constraints, can save in a riskless asset. The proof itself is rather straightforward and relies on the aggregation of individual Euler equations in a stationary equilibrium for general utility functions. For the sake of conciseness, the core of the paper focuses on a finite-state idiosyncratic risk. We also discuss several extensions, including a continuous space for the idiosyncratic risk, a more general income process, and capital taxes. These extensions do not affect our results – including non-existence. Overall, our characterization of the relationship between the interest rate and credit constraints is very general and holds as long as households have access to an asset whose return is not affected by their idiosyncratic risk.

This paper is written as a note that focuses on characterizing the interest rate and the existence of stationary equilibria, using a simple proof in a general environment. To the best of our knowledge, the result about the interest rate is new. The result about existence characterization clarifies a rather vast literature on equilibrium existence, which has been studied in various specific environments. Krebs (2004), for instance, shows that a recursive equilibrium cannot exist in a two-agent incomplete-market economy – without production – when credit constraints are non-binding. Several technical assumptions must also hold: the per period utility function must be unbounded from below (which rules out CARA utilities and CRRA utilities with an elasticity of substitution above 1, for instance) and endowment processes must follow a Markov chain with finite support. We consider a standard heterogeneous-agent economy (with a continuum of agents), with a standard utility function (strictly increasing, concave) and production. Our result holds for any stationary equilibrium (independent of a recursive or sequential formulation) and can easily be extended to a large class of income processes. Furthermore, we also provide a robust relationship between the interest rate and credit constraints in any stationary equilibrium economy.

Another related paper is that of Miao (2002), who studies the existence of a stationary recursive competitive equilibrium in a heterogeneous-agent economy with production. Miao's analysis, however, relies on certain assumptions (smoothness condition on the Markov process, upper bound on the utility function, curvature assumption on the utility function). We relax all of these assumptions and further show that any stationary equilibrium, if it exists, must feature binding credit constraints for a positive mass of agents whenever individual income is stochastic (in the spirit of Chamberlain and Wilson 2000).

Finally, Açikgöz (2018) provides existence results for stationary equilibria when credit constraints bind in equilibrium. Our analysis shows that this appears to be a complete characterization of stationary equilibria in standard incomplete market economies.

More generally, our paper is also complementary to the long-standing strand of the literature proving equilibrium existence in heterogeneous-agent models, that started with Bewley (1986) in a monetary economy. Huggett (1993) proves existence of a stationary equilibrium in an endowment economy, assuming a two-state monotone Markov process. Zhu (2017) proves existence in a production economy with endogenous labor supply, but with bounded utility functions. Kuhn (2013) relaxes the boundedness assumption but only considers CRRA utility function and an endowment economy with mortality and IID and permanent income shocks. Acemoglu and Jensen (2015) prove equilibrium existence for an exogenously bounded set of asset choices in a set-up that is not restricted to Bewley economies.[1] Finally, a last set of papers, namely Miao (2006), Cheridito and Sagredo (2016), and Cao (2020), prove, under various technical restrictions, the existence of a competitive sequential equilibrium in incomplete-market economies with aggregate shocks.

The rest of the article is organized as follows. Section 2 presents the environment.

[1]Our result is derived in the standard case that does not feature such bounds on individual saving choices. It would be relatively straightforward to extend the results to their setup, in which the equilibrium interest rate would depend on the mass of agents at the upper bound of the choice set.

Section 3 states our main characterization result and provides its proof in the case of a sequential formulation and finite-state idiosyncratic risks. Section 4 presents several extensions, while Section 5 discusses the implications for the literature.

2 Environment

The environment is a standard heterogeneous-agent economy with production in discrete time. We assume that the economy is populated by a continuum of agents with unit mass, and distributed on an interval $\mathcal{I}$ according to a measure $\ell(\cdot)$. We follow Green (1994) and assume that the law of large numbers holds.[2]

2.1 Idiosyncratic risk structure

In each period, every agent inelastically supplies one unit of labor and receives an income in exchange for her labor supply. This income, denoted by e_t, is risky and this risk cannot be insured or avoided. Income realizations belong to a set denoted by E that is assumed to be finite and to contain only distinct possible income realizations.[3] Furthermore, the income process follows a finite-state first-order Markov chain and we denote by $\Pi_{e,e'}$ the (constant) probability of switching from the current income $e \in E$ to the income $e' \in E$ in the next period.

The history of idiosyncratic shocks from date 0 to date t is denoted by $e^t = \{e_0, \ldots, e_t\} \in E^{t+1}$ and gathers all shock realizations prior to date t. We denote by $e^{t+1} \succeq e^t$ the fact that history e^{t+1} is a possible continuation of history e^t, meaning that the realizations of e^t and e^{t+1} coincide for all dates from 0 to t. Finally, the quantity $\tilde{\Pi}_{e^t,e^{t+1}}$ represents the probability of switching from history e^t at t to history e^{t+1} at $t+1$. This probability is equal to the transition probability from e_t at t to e_{t+1} at $t+1$ if e^{t+1} is a continuation of e^t. Formally: $\tilde{\Pi}_{e^t,e^{t+1}} \equiv \Pi_{e_t,e_{t+1}} 1_{e^{t+1} \succeq e^t}$, where $1_{e^{t+1} \succeq e^t} = 1$ if $e^{t+1} \succeq e^t$ and 0 otherwise.

We allow the initial distribution of agents (at date 0) to depend on the agent's initial income. This assumption offers generality and makes it possible for the economy to start from the steady-state wealth distribution. We denote by $A = [-\underline{a}, \infty)$ the set of possible wealth levels. Agents are prevented from borrowing more than $\underline{a} \geq 0$, which bounds the set A from below. Note that this borrowing limit – as for any finite limit – could equivalently be set to zero without loss of generality after a proper renormalization of the income process (see Aiyagari 1994). Zero is the natural borrowing limit if productivity is null for one idiosyncratic state, for instance.[4]

[2]See also Miao (2006) for a careful treatment of the law of large numbers in these economies.

[3]See Section 4 and Appendix B for extensions of this simple framework, for instance to the case of an uncountable set E.

[4]The natural borrowing limit is the smaller borrowing limit (in absolute value) such that credit constraints do not bind in equilibrium. It is called the "present value" borrowing limit in Aiyagari

We denote by $\mathcal{A}$ the σ-algebra of Borel sets of A. Similarly, we denote by $\mathcal{E}$ the power set of E.[5] We assume that the initial distribution of agents at date 0 is characterized by the measure μ_0 defined on the product σ-algebra $\mathcal{E} \times \mathcal{A}$, such that for any $A_0 \in \mathcal{A}$ and $E_0 \in \mathcal{E}$, $\mu_0(E_0, A_0)$ is the measure of agents with initial wealth in A_0 and history in E_0. Note that, with a slight abuse of notation, we will denote the measure of agents with initial income e_0 and initial wealth in A_0 by $\mu_0(e_0, A_0)$, instead of $\mu_0(\{e_0\}, A_0)$. The distribution of agents at future dates $t \geq 1$ will depend on idiosyncratic history evolution and on initial wealth – since, loosely speaking, the initial dependence at date 0 will propagate at "later" dates. Noting that $\mathcal{E}^t$ is the σ-algebra defined on the product space E^t, we will denote by $\mu_t\left(E_t, A_0\right)$ the measure of agents with idiosyncratic history $e^t \in E_t$ at date t and initial wealth $a_0 \in A_0$. First, note that since the total measure of the population is constant and equal to 1 at all dates, we have, at all dates t:

$$\int_{a_0 \in A} \sum_{e^t \in E^t} \mu_t(e^t, da_0) = 1.$$

Second, using Bayes' law, the measure μ_{t+1} can be expressed using the measure μ_t and transition probabilities $(\tilde{\Pi}_{e^t,e^{t+1}})_{e^t,e^{t+1}}$. Formally, for any $e^{t+1} \in E^{t+1}$ and any $A_0 \in \mathcal{A}$:

$$\mu_{t+1}(e^{t+1}, A_0) = \sum_{e^t \in E^t} \tilde{\Pi}_{e^t,e^{t+1}} \mu_t(e^t, A_0). \tag{1}$$

2.2 Agents' program

Agents are expected-utility maximizers with standard time-additive preferences. The discount factor $\beta \in (0, 1)$ is constant and the period utility function, denoted by $u : \mathbb{R}_+ \to \mathbb{R}$, is twice continuously differentiable, increasing, and strictly concave.

Agents can transfer resources from one period to another through a security (capital shares) that pays off the deterministic interest rate r. However, as already explained, borrowing is limited and agents cannot borrow more than the amount $\underline{a} \geq 0$. Agents choose their consumption path $(c_t)_{t\geq 0}$ and their saving path $(a_{t+1})_{t\geq 0}$ so as to maximize their expected utility, subject to credit limits and the borrowing constraint. The latter states that, at any date, spending on consumption and savings cannot exceed resources, comprising savings payoffs and income. Formally, the agent's program can be expressed

(1994).

[5]Note that $\mathcal{E}$ can also be seen as the Borel sets of the discrete space E endowed with discrete topology.

as:

$$\max_{\{c_t,a_{t+1}\}_{t=0}^{\infty}} \mathbb{E}\sum_{t=0}^{\infty}\beta^t u\left(c_t\right), \tag{2}$$

$$\text{s.t. } c_t + a_{t+1} \leq (1+r)\, a_t + e_t, \tag{3}$$

$$a_{t+1} \geq -\underline{a}, \tag{4}$$

$$a_0, e_0 \text{ given.} \tag{5}$$

In equation (2), the unconditional expectation $\mathbb{E}[\cdot]$ is taken over the future income stream, which is the only stochastic variable. A solution to the household problem (2)–(5) is a sequence of measurable consumption functions $c_t : E^t \times A \to \mathbb{R}^+$ and a sequence of measurable Lagrange multipliers on the credit constraint $\nu_t : E^t \times A \to \mathbb{R}^+$, solving the standard Euler equation at all dates t:

$$u'\left(c_t(e^t, a_0)\right) = \beta\,(1+r) \sum_{e^{t+1}\in E^{t+1}} \tilde{\Pi}_{e^t,e^{t+1}} u'\left(c_{t+1}(e^{t+1}, a_0)\right) + \nu_t(e^t, a_0), \tag{6}$$

where $\sum_{e^{t+1}\in E^{t+1}} \tilde{\Pi}_{e^t,e^{t+1}}[\cdot]$ is the conditional expectation operator written in explicit form. Note that when the credit constraint does not bind at date t for initial wealth a_0 and history e^t, we have $\nu_t(e^t, a_0) = 0$. The quantity $\nu_t(e^t, a_0)$ can be interpreted as the shadow price of the agent's credit constraint (4). The saving functions $a_{t+1} : E^t \times A \to \mathbb{R}^+$ can then be deduced from the budget constraints.

2.3 Production

In each period t, a representative firm produces output Y_t using capital K_t and labor L_t. The firm rents the capital at a rate r_t and the labor at a wage w_t. Production net of depreciation is $F\left(K_t, L_t\right) - \delta K_t$ where $\delta \in (0,1)$ is the depreciation rate and $F(\cdot,\cdot)$ is a constant-returns-to-scale production function, strictly increasing and concave. Profit maximization implies (F_K and F_L denote the partial derivatives):

$$r_t = F_K\left(K_t, L_t\right) - \delta, \quad w_t = F_L\left(K_t, L_t\right). \tag{7}$$

Capital and total labor supply are defined by market clearing conditions:

$$K_t = \int_{a_0\in A} \sum_{e^t\in E^t} a_t(e^t, da_0)\mu_t(e^t, da_0), \qquad L_t = \int_{a_0\in A} \sum_{e^t\in E^t} e_t \mu_t(e^t, da_0). \tag{8}$$

2.4 Equilibrium

We now define the concept of stationary competitive equilibrium.

Definition 1 (Equilibrium) *A stationary competitive equilibrium is a collection of individual allocations* $(c_t^i, a_t^i)_{t\geq 0, i\in\mathcal{I}}$*, aggregate quantities* (K, L)*, and price processes* (w, r)*,* *such that, for an initial wealth distribution $(a_0^i)_{i\in\mathcal{I}}$, we have:*

1. *For given prices, individual allocations $(c_t^i, a_t^i)_{t\geq 0, i\in\mathcal{I}}$ solve the agent's program (2)–(5);*

2. *Financial, labor, and goods markets clear at all dates. Finite values $K, L \in \mathbb{R}$ exist such that for any $t \geq 0$:*

$$K = \int_{a_0\in A} \sum_{e^t\in E^t} a_t(e^t, da_0)\mu_t(e^t, da_0), \qquad L = \int_{a_0\in A} \sum_{e^t\in E^t} e_t\mu_t(e^t, da_0),$$
$$F(K, L) = \int_{a_0\in A} \sum_{e^t\in E^t} c_t(e^t, da_0)\mu_t(e^t, da_0) + \delta K;$$

3. *Factor prices are consistent with (7): $r = F_K(K, L) - \delta$ and $w = F_L(K, L)$;*

4. *The distribution of marginal utilities is constant (and finite) over time, i.e.:*

$$\int_{a_0\in A} \sum_{e^t\in E^t} u'\left(c_t(e^t, da_0)\right)\mu_t(e^t, da_0) = \int_{a_0\in A} \sum_{e^{t+1}\in E^{t+1}} u'\left(c_{t+1}(e^t, da_0)\right)\mu_{t+1}(e^t, da_0). \quad (9)$$

In our stationary equilibrium definition, points 1 to 3 are very standard – see Açikgöz (2018) for instance. We focus on the equilibrium with constant prices and constant aggregate quantities. Point 4 is however weaker than in the standard formulation, which usually assumes that the distribution of asset holdings is constant in the economy. We only require the aggregation of individual marginal utilities to be constant.

3 Main result

The following proposition contains our main result.

Proposition 1 (Interest rate) *In any existing stationary equilibrium, the interest rate must satisfy:*

$$\frac{1}{\beta} - (1+r) = \frac{\int_{a_0} \sum_{e^t\in E^t} \nu(e^t, a_0)\mu_t(e^t, da_0)}{\beta \int_{a_0} \sum_{e^t\in E^t} u'\left(c_t(e^t, a_0)\right)\mu_t(e^t, da_0)}. \quad (10)$$

As the proof is short, we first provide it before discussing the Proposition.

Proof. Aggregating the Euler equations (6) over all possible histories $e^t \in E^t$ and all initial asset holdings $a_0 \in A$ yields:

$$\int_{a_0} \sum_{e^t\in E^t} u'\left(c_t(e^t, a_0)\right)\mu_t(e^t, da_0) - \int_{a_0} \sum_{e^t\in E^t} \nu(e^t, a_0)\mu_t(e^t, da_0)$$
$$= \beta(1+r) \int_{a_0} \sum_{e^t\in E^t} \sum_{e^{t+1}\in E^{t+1}} \Pi_{e^t, e^{t+1}} u'\left(c_{t+1}(e^{t+1}, a_0)\right)\mu_t(e^t, da_0),$$
$$= \beta(1+r) \int_{a_0} \sum_{e^{t+1}\in E^{t+1}} u'\left(c_{t+1}(e^{t+1}, a_0)\right)\left(\sum_{e^t\in E^t} \Pi_{e^t, e^{t+1}}\mu_t(e^t, da_0)\right),$$

where the last equality comes from the permutation of the two finite sums. Using the recursive definition of (μ_t) in equation (1) stating that the term between brackets is μ_{t+1}, our stationarity property (9) readily implies the expression (10). Note that we can rule out $\int_{a_0} \sum_{e^t \in E^t} u'\left(c_t(e^t, a_0)\right) \mu_t(e^t, da_0) = 0$ for some t. Indeed, should it hold, this would imply $u'\left(c_t(e^t, a_0)\right) = 0$ almost surely and thus $c_t(e^t, a_0) = \infty$ almost surely (because $u' > 0$ and $u'' < 0$), which is not compatible with a stationary equilibrium of a finite economy. ■

Proposition 1 states that the gap between the discount rate and the interest rate is proportional to the average shadow price of credit constraints, where the average is computed over all possible idiosyncratic histories and initial asset holdings. This equilibrium outcome can be seen from two perspectives. First, if the interest rate $1 + r$ is below the discount rate $1/\beta$, then self-insurance is costly. As a consequence, households rationally choose not to perfectly self-insure themselves and hit the credit limit with probability 1 in some states of the world. This is an important step in the proof of existence presented by Açikgöz (2018), for instance. Conversely, when the credit constraint binds in some states of the world, households want to save to transfer resources to this state of the world. As a consequence, they accept a lower return, relative to the complete market economy, due to this self-insurance motive. This generates a liquidity premium on the asset.[6]

For the sake of simplicity, Proposition 1 and its proof are stated with a sequential formulation of the model. The proposition also holds using a recursive formulation (see Appendix A).

The following corollary is immediate, as it is a reformulation of Proposition 1.

Corollary 1 (Stationary equilibrium characterization) *Any existing stationary equilibrium must feature either:*

- $\beta\,(1 + r) < 1$ *and binding credit constraints for a positive measure of agents; or*
- $\beta\,(1 + r) = 1$ *and non-binding credit constraints (almost surely).*

Corollary 1 provides a straightforward characterization of any stationary equilibrium (whenever it exists).

We conclude this section with a very general impossibility result.

Corollary 2 (Existence) *If* $\mathrm{Card}\, E \geq 2$ *and if for all* $e, e' \in E$, $\Pi_{ee'} \in (0, 1)$*, a stationary equilibrium cannot exist when credit constraints are non-binding for a positive mass of agents.*

Proof. The corollary is the direct consequence of Proposition 1. It is proven by contradiction. If a stationary equilibrium exists and credit constraints do not bind, we must

[6] Liquidity is defined here as an asset's ability to transfer some wealth in the state of the world where the credit constraint binds.

have $\beta(1+r)=1$, which has been shown to be incompatible with the existence of a stationary equilibrium. More precisely, Chamberlain and Wilson (2000) (Corollary 2, p. 381) show under very general conditions that if the discounted income stream has sufficient variability, the consumption path will almost surely diverge to infinity. In our case, with a stationary Markovian process, the variability condition implies that $\mathbb{V}_0\left[\sum_{t=0}^{\infty}\beta^t e_t|e_0\right]$ is bounded away from 0 for any initial state e_0. Since $\beta>0$, $\mathbb{V}_0\left[\sum_{t=0}^{\infty}\beta^t e_t|e_0\right]=0$ implies either a unique income level (and thus no income risk) or a transition matrix $(\Pi_{e'e})$ containing an attractive state. The first point is made impossible by $\operatorname{Card} E\geq 2$ and the second one by $\Pi_{ee'}\in(0,1)$ for all (e,e'). ■

Corollary 1 states that any stationary equilibrium – whenever it exists – must feature a binding credit constraint for a positive measure of agents, as long as income is volatile. In the case of the finite-state Markovian process, this only rules out polar cases: one-state Markov chains and conditionally deterministic Markov chains (i.e., with transition matrices containing only zeros and ones).

A numerical example. For illustrative purposes, we simulate a standard incomplete market economy. We plot the asset demand for varying credit limits. Agents are assumed to have a labor endowment equal to $e_1=1$ or $e_2=0.8$. The transition matrix across these two idiosyncratic states is assumed to be symmetrical.

$$\Pi=\begin{bmatrix}0.9 & 0.1\\ 0.1 & 0.9\end{bmatrix}.$$

The discount factor is $\beta=0.96$, the production function is $F(K,L)=K^\alpha L^{1-\alpha}$, with $\alpha=1/3$, and the depreciation rate is $\delta=0.1$. We then compute household asset demand as a function of the interest rate r. We perform this exercise in two environments. In the first one, the credit limit is set to $\underline{a}=0$. In the second, the credit limit is close to the natural borrowing constraint $\underline{a}^n(r)=-(1-\varepsilon)we_1/(1+r)$, where $\varepsilon=1\%$. In other words, the credit limit is 99% of the natural borrowing limit.

Figure 1 plots our results using the Aiyagari (1994) representation. The x-axis corresponds to asset quantity and the y-axis to the interest rate. The black solid line is the firm's capital demand, which is steadily downward sloping (and looks vertical due to the scale of the x-axis). The horizontal black dashed line plots the interest rate $1+r=1/\beta$, which would prevail in the complete market economy. The red line plots household asset demand in the economy where $\underline{a}=0$. The blue solid line plots asset demand when borrowing is set to $\underline{a}^n(r)$, which varies with the interest rate r.

A looser credit limit translates the asset demand to the left. This result is derived theoretically by Acemoglu and Jensen (2015). A tightening of the credit limit, namely a "positive shock" using their wording, increases household savings through the self-insurance motive. The equilibrium interest rate in these two economies is at the inter-

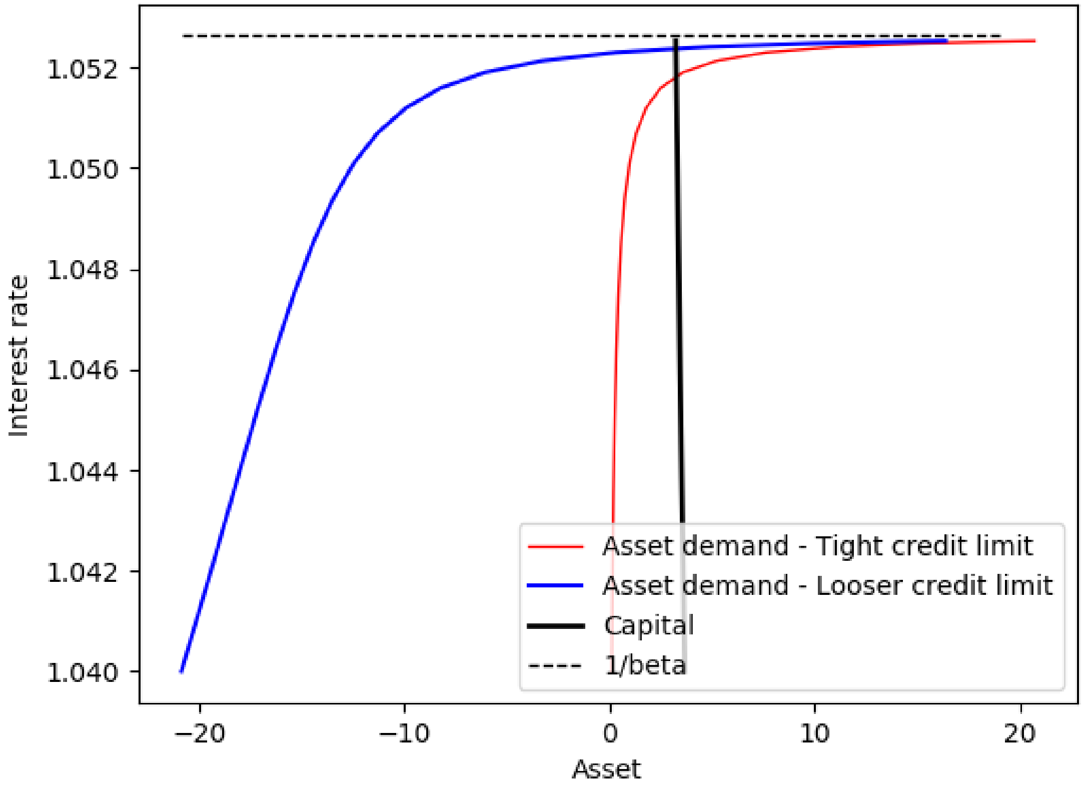

Figure 1: Asset supply and demand for different credit limits

section between the black solid line and the relevant asset-demand curve. The severity of the credit limit, measured as the right-hand side of equation (10), is the difference between the equilibrium interest rate and the dashed line representing $1/\beta$.

4 Extensions

In this section, we briefly discuss various extensions to the result of Proposition 1, as well as those of Corollaries 1 and 2 (besides the recursive formulation of Appendix A).

A continuous income space. Generalizing the income space E to a continuous space – while maintaining the Markovian structure – is rather straightforward and does not affect any of our results. The proof is also roughly the same, with the only major difference being a technicality: the discrete sum over idiosyncratic histories becomes an integral. See Appendix B for a formal presentation.[7]

The results, in particular Proposition 1 and Corollary 1, would also hold for much more

[7]Obviously, the σ-algebra $\mathcal{E}$ on E is now the Borel algebra of E and not the power set of E. The measure μ_t defined on $\mathcal{E}^t \times \mathcal{A}$ is thus changed accordingly.

general income processes (including non-Markovian ones). As in Chamberlain and Wilson (2000), the non-existence result requires sufficient variability in the income process.

Capital tax. Proposition 1, and Corollaries 1 and 2, still hold if a linear capital tax is allowed. The only difference is that the results will be formulated in terms of the post-tax interest rate (rather than the pre-tax one). Households face the post-tax rate and it is thus this rate that matters for their decisions.

Endogenous labor supply. Introducing endogenous labor supply (with idiosyncratic productivity risk) will not affect the results of Proposition 1 and Corollary 1. Indeed, the individual Euler equations for consumption remain valid, even though they may also depend on labor choices. Their aggregation still leads to the characterization of the interest rate at any stationary equilibrium.

The existence result of Corollary 2 will, however, depend on the formalization of the labor supply. Marcet, Obiols-Homs, and Weil (2007) have shown that a stationary equilibrium can exist when $\beta(1 + r) = 1$ if the wealth effect on the labor supply is sufficiently high to reduce both labor income and the capital accumulation of wealth-rich agents. Conversely, if the wealth effect is low or absent – as in the case of a Greenwood-Hercowitz-Huffman utility function – then no stationary equilibrium exists and Corollary 2 holds.

Summary. In short, Proposition 1 and Corollary 1 are robust to various extensions and will hold as long as the two following key conditions are present.

1. There should be at least one asset whose interest rate is not affected by agents' idiosyncratic risk. In loose terms, the interest rate needs to be "taken out" of their idiosyncratic risk expectation.

2. Aggregating individual Euler equations should generate the aggregate values of marginal utilities for the whole population. This notably implies the existence of a stationary distribution and also implies that in the presence of mortality (as in the Blanchard-Yaari model) some specific assumptions about the initial endowments of new-born agents need to be made to maintain our result.

5 Implications for the literature

The above analysis has important implications for the literature. First, the seminal paper of Aiyagari (1994) focuses on stationary equilibria in various contexts, including the case of non-binding credit constraints. In some analyses, the borrowing limit is then set to a value lower than the so-called "present value" borrowing limit, or the natural borrowing limit. One of the conclusions of our paper is that Ricardian equivalence holds in this environment. Unfortunately, however, the stationary equilibrium does not exist and the long-run effect of public debt in stationary environments should be analyzed in the presence of binding credit constraints.

Second, Aiyagari (1995) analyzes optimal capital and labor taxes, and public debt, in a standard incomplete-market model, where the government has to finance public consumption. Aiyagari finds that, in this economy, the government sets the before-tax interest rate $\tilde{r}$ equal to the discount rate $1/\beta - 1$, or equivalently: $\beta(1 + \tilde{r}) = 1$. This is called the "Golden Rule". As no stationary equilibrium can exist when $\beta(1 + r) = 1$ (r again denoting the post-tax interest rate faced by agents), he concludes that capital taxes must be positive for any stationary equilibrium to exist. Our finding confirms this statement. More precisely, our previous analysis shows that the government implicitly chooses the "measure" of agents who are credit-constrained. This property could help prove that stationary equilibria with optimal positive capital tax do indeed exist, which, to the best of our knowledge, has not yet been achieved.

Third, the seminal papers of Angeletos and Calvet (2005, 2006) consider non-binding credit constraints in the CARA-normal case in an economy where households are entrepreneurs endowed with their own production function and facing an idiosyncratic production risk (either through the total productivity factor or the depreciation rate). Their results are derived in a non-stationary equilibrium, where the variance of consumption is unbounded. Our non-existence result, which applies to their framework, confirms that focusing on a stationary equilibrium is not possible in such setups.

Finally, a recent literature stream is currently analyzing optimal Ramsey policies in incomplete market economies. A potential method for deriving the planner's first-order conditions is the "primal approach", which is often used in complete market economies (see Chari, Nicolini, and Teles 2019 for a recent discussion and formulation). This approach uses households' non-binding Euler equations as a substitute for the real interest rate. As shown by our analysis, no stationary equilibrium exists in this case, which limits the scope of this approach to non-stationary economies. An alternative strategy is to use the "Lagrangian approach" developed by Marcet and Marimon (2019). Some progress in this direction has been made by Açikgöz, Hagedorn, Holter, and Wang (2018) and Le Grand and Ragot (2019).

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Appendix

A The proof using a recursive representation

In the interests of clarity, the paper's setup and proofs were provided using a sequential representation. We prove here that results still hold in a recursive representation. As we provide a characterization of the interest rate and existence in any stationary equilibrium, we will simply assume that it exists, and derive proofs by contradiction. The framework is the same as that described in Section 2 – in particular the income space E is discrete.

We assume that a stationary probability measure $\psi : \mathcal{E} \otimes \mathcal{A} \rightarrow [0,1]$ exists such that $\psi(e, A_0)$ is the stationary measure of agents of productivity type $e \in E$, who hold a quantity of assets in the set $A_0 \subset A$. The agents' program written in recursive form is defined by the value function $V(e,a) = max_{c,a'} u(c) + \beta \mathbb{E} V(e', a')$, subject to the budget constraint $a' + c = a(1+r) + we$ and to the credit constraint $a' \geq -\underline{a}$.

A stationary equilibrium is defined as a set of policy functions $c(e,a)$ and $a' = g_a(e,a)$, for consumption and savings respectively, prices r, w, and a stationary distribution ψ such that: (i) the policy functions solve the agents' program when prices are given; (ii) capital and labor markets clear: $K = \sum_{e \in E} \int_A g_a(e,a) \, d\psi(e, da)$ and $L = \sum_{e \in E} \int_A e d\psi(e, da)$; and (iii) ψ is invariant because of the transition functions generated by the policy rules and the law of motions of the income space – which formally means that, for all $A_0 \in \mathcal{A}$ and $e \in E$:

$$\psi(e, A_0) = \sum_{\tilde{e} \in E} \int_{a \in A} 1_{A_0}(g_a(\tilde{e}, a)) \, \Pi_{\tilde{e},e} \psi(\tilde{e}, da),$$

where $1_{A_0}(g_a(a, \tilde{e})) = 1$ if $g_a(a, \tilde{e}) \in A_0$ and 0 otherwise. The first-order condition for the agent's program can be written as:

$$u'(c(e,a)) = \beta(1+r) \sum\nolimits_{e' \in E} \Pi_{e,e'} u'(c(e', g_a(e,a))) + \nu(e,a), \tag{11}$$

where $\nu(e,a)$ is the Lagrange multiplier on the credit constraint.

In such an equilibrium, the distribution of marginal utilities is constant over time, as is the average marginal utility in the economy. Seen from the current period, the next period average marginal utility is given by the policy rules and the law of motion for the productivity shock. Indeed, for agents with a productivity level e and a wealth level $a \in A$, their next period marginal utility, if they happen to have productivity $e' \in E$, is given by $u'(c(e', a')) = u'(c(e', g_a(e,a)))$ and there will be a fraction $\Pi_{e,e'} \times \psi(e, da)$ of such agents. As a consequence, the next period average marginal utility in the economy being equal to that of the current period, we have:

$$\sum_{e,e' \in E} \int_A u'(c(e', g_a(e,a))) \, \Pi_{e,e'} \psi(e, da) = \sum_{e \in E} \int_A u'(c(e,a)) \, \psi(e, da). \tag{12}$$

Integrating the Euler equation (11), we deduce that:

$$\sum_{e\in E}\int_A u'\left(c\left(e,a\right)\right)\psi\left(e,da\right)=\beta\left(1+r\right)\sum_{e\in E}\int_A \sum\nolimits_{e'\in E}\Pi_{e,e'}u'\left(c\left(e',g_a(e,a)\right)\right)\psi\left(e,da\right)$$
$$+\sum_{e\in E}\int_A \nu\left(e,a\right)\psi\left(e,da\right).$$

Using the stationarity of the average marginal utility in the economy of equation (12), we can directly state the following proposition.

Proposition 2 (Interest rate) *If a recursive stationary equilibrium exists, then:*

$$1+r=\frac{1}{\beta}-\frac{\sum_{e\in E}\int_A \nu\left(e,a\right)\psi\left(e,da\right)}{\beta\sum_{e\in E}\int_A u'\left(c\left(e,a\right)\right)\psi\left(e,da\right)}.$$

Proposition 2 is the strict parallel of Proposition 1 for the recursive formulation. As a consequence, when credit constraints do not bind, we again have $\beta(1+r)=1$ for any existing stationary equilibrium, and Corollaries 1 and 2 still hold.

B Proof for a continuous income state-space

We now provide the proof for a continuous income space $E\subset\mathbb{R}^+$ and its Borel algebra $\mathcal{E}$. We consider a transition kernel p, which extends the notion of transition probabilities. More precisely, for any $e\in E$ and any $E_0\in\mathcal{E}$, $p(E_0|e)$ is the probability of reaching an income $e'\in E_0$ from the income e. From p, we can deduce the transition kernel for histories, denoted $\tilde{p}$, which is defined by: $\tilde{p}\left(\prod_{\tau=0}^{t+1}E_\tau\,|e^t\right)=\int_{e^{t+1}\in\prod_{\tau=0}^{t+1}E_\tau}1_{e^{t+1}\succeq e^t}p(de_{t+1}|e_t)$ for all $\prod_{\tau=0}^{t+1}E_\tau\in\mathcal{E}^{t+1}$ and all $e^t\in E^t$. Note that making the relationships between e_t and e^t explicit ($e^t=(\tilde{e}^{t-1},e_t)$) and between e_{t+1} and e^{t+1} explicit ($e^{t+1}=(\tilde{e}^t,e_{t+1})$), the definition of $\tilde{p}$ can also be written as: $\tilde{p}\left(\prod_{\tau=0}^{t+1}E_\tau\,|(\tilde{e}^{t-1},e_t)\right)=\int_{(\tilde{e}^t,e_{t+1})\in\prod_{\tau=0}^{t+1}E_\tau}1_{\tilde{e}^t=e^t}p(de_{t+1}|e_t)$.

As in the main text, we start from an initial distribution μ_0 defined over $\mathcal{E}\times\mathcal{A}$, such that $\int_{E\times A}\mu_0(de_0,da_0)=1$. The distribution at date t, denoted by μ_t is defined over $\mathcal{E}^t\times\mathcal{A}$ and verifies the following recursion:

$$\mu_{t+1}\left(\prod_{\tau=0}^{t+1}E_\tau,A_0\right)=\int_{(e^{t+1},a_0)\in\prod_{\tau=0}^{t+1}E_\tau\times A_0}\int_{e^t\in E^t}\tilde{p}(de^{t+1}|e^t)\mu_t(de^t,da_0),\tag{13}$$

where $E_\tau\in\mathcal{E}$ and $A_0\in\mathcal{A}$. Using infinitesimal notation, we can alternatively write μ_{t+1}:

$$\mu_{t+1}(de^{t+1},da_0)=\int_{e^t\in E^t}\tilde{p}(de^{t+1}|e^t)\mu_t(de^t,da_0).\tag{14}$$

We can now state a result similar to that of Proposition 1.

Proposition 3 (Interest rate) *In any existing stationary equilibrium, the interest rate has to satisfy:*

$$1 + r = \frac{1}{\beta} - \frac{\int_{E^t \times A} \nu(e^t, a_0)\mu_t(de^t, da_0)}{\beta \int_{E^t \times A} u'\left(c_t(e^t, a_0)\right)\mu_t(de^t, da_0)}. \tag{15}$$

Proof. The Euler equation of the agent's program can be written as:

$$u'\left(c_t(e^t, a_0)\right) = \beta\,(1+r) \int_{e^{t+1} \in E^{t+1}} u'\left(c_{t+1}(e^{t+1}, a_0)\right) \tilde{p}(de^{t+1}|e^t) + \nu(e^t, a_0) \tag{16}$$

Integrating over the whole population, the set of Euler equations (16) over the whole distribution of agents yields:

$$\int_{(e^t, a_0) \in E^t \times A} u'\left(c_t(e^t, a_0)\right)\mu_t(de^t, da_0) = \int_{(e^t, a_0) \in E^t \times A} \nu(e^t, a_0)\mu_t(de^t, da_0)$$
$$+\beta\,(1+r) \int_{(e^t, a_0) \in E^t \times A} \int_{e^{t+1} \in E^{t+1}} u'\left(c_{t+1}(e^{t+1}, a_0)\right)\tilde{p}(de^{t+1}|e^t)\mu_t(de^t, da_0).$$

Since $(e^{t+1}, a_0) \mapsto u'\left(c_{t+1}(e^{t+1}, a_0)\right)$ is assumed to be Lebesgue integrable, Fubini's theorem yields:

$$\int_{(e^t, a_0) \in E^t \times A} u'\left(c_t(e^t, a_0)\right)\mu_t(de^t, da_0) = \int_{(e^t, a_0) \in E^t \times A} \nu(e^t, a_0)\mu_t(de^t, da_0)$$
$$\beta\,(1+r) \int_{(e^{t+1}, a_0) \in E^{t+1} \times A} u'\left(c_{t+1}(e^{t+1}, a_0)\right) \int_{e^t \in E^t} \tilde{p}(de^{t+1}|e^t)\mu_t(de^t, da_0),$$

or using the recursive definition (14) of μ_{t+1}:

$$\int_{E^t \times A} u'\left(c_t(e^t, a_0)\right)\mu_t(de^t, da_0) = \beta\,(1+r) \int_{E^{t+1} \times A} u'\left(c_{t+1}(e^{t+1}, a_0)\right)\mu_{t+1}(de^{t+1}, da_0)$$
$$+ \int_{E^t \times A} \nu(e^t, a_0)\mu_t(de^t, da_0)$$

Using stationarity, stated similarly to equation (9), then implies equality (15). ■

Loosely speaking, introducing a continuous state-space only changes the integral, which is now a continuous sum over histories, while it is a discrete sum in the main text. Proposition 3 is thus very similar to Proposition 1, and Corollaries 1 and 2 still hold.