LIMITED PARTICIPATION, CAPITAL ACCUMULATION AND OPTIMAL MONETARY POLICY

XAVIER RAGOT

ABSTRACT. Motivated by recent empirical findings on money demand, the paper presents a general equilibrium model where agents have limited participation in financial markets and use money to smooth consumption. In such setup, investment is not optimal because only a fraction of households participate in financial markets in each period. Optimal monetary policy substantially increases welfare by changing investment decisions over the business cycle, but adverse redistributive effects limit the scope for an active monetary policy. Recent developments in the heterogeneous-agents literature are used to develop a tractable framework with aggregate shocks, where optimal monetary policy can be analyzed.

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The rapid expansion of central bank balance sheets in the US, Japan and the Euro area after the 2008 crisis has rejuvenated old but deep questions: What are the real effects of money injections? Do they affect investment and economic activity ? Should money creation be used to affect investment dynamics ? For policy makers, the link between monetary policy and investment is explicit. The Federal Reserve Board actually produces a systematic assessment of business investment when presenting monetary policy decisions. Reading the minutes of the Fed, one observes that the prospects of business fixed investment are discussed in the process of policy making.

The goal of this paper is to analyze the conditions under which optimal monetary policy should consider investment, in a simple but micro-founded monetary model. It develops

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Xavier Ragot, SciencesPo CNRS (UMR 8259) and OFCE. Email: xavier.ragot@gmail.com.

a tractable model where agents have limited participation in financial markets and face aggregate shocks. In this setup, agents use money to smooth consumption between periods at which they adjust their financial portfolio. The focus on limited participation as the key friction comes from two sets of results. First, the distribution of money across households generated by this friction is much more similar to the data than the distribution generated by alternative money demand (Alvarez and Lippi, 2009; Cao, Meh, Rios-Rull and Terajima, 2012; Ragot, 2014). Reproducing a relevant money distribution is key to assess the redistributive effect of monetary policy. Second, Kaplan, Violante and Weidner (2014) and Kaplan and Violante (2014) have shown that introducing limited participation allows to better reproduce the empirical consumption/saving choice after a fiscal transfer. Thus both the incentives to save and to hold money can be captured by this simple friction. The main result of the paper is that this friction implies that investment is not optimal and that monetary policy should indeed restore the right incentives to save over the business cycle. It thus provides a rationale for monetary policy to consider investment over the business cycle.

The paper first presents a simple general-equilibrium model to derive formal proofs. Then, it provides a more general framework to study optimal monetary policy with a simplified but realistic money distribution. It is already known that limited participation generates some relevant sort-run effects, such as the liquidity effect of money injection: An increase in the quantity of money decreases the nominal interest rate, as only a part of the population must absorb the new money created (Lucas, 1990; Alvarez, Atkeson and Edmond, 2009, among others). Nevertheless, this promising literature has faced some difficulties in dealing with agents' heterogeneity (see the literature review below). This has prevented the introduction of additional features which are important for understanding the business cycle, such as long-lasting heterogeneity, aggregate shock and capital accumulation. Developments of tractable environments in the heterogeneous agent literature allow deriving new results about optimal monetary policy in these economies.

Analyzing the simple model, one first finds that the distortions generated by limited participation are surprisingly not simple. In general, capital accumulation is not optimal, as a part of the income generated by the capital stock is distributed as wages to households who do not participate in financial markets. The direction of the distortions (for instance the over or under accumulation of capital after a technology shock) crucially depends on the persistence of the technology shock, because of income and substitution effects. Money creation can restore the first-best allocation by affecting capital accumulation. For instance, to increase aggregate saving, money creation induces a transfer between non-participating and participating households, which also implements optimal consumption levels for all agents. In addition, the optimal allocation cannot be implemented by a time-varying capital tax, because it would distort the intertemporal consumption smoothing. In this sense, monetary policy is a powerful tool to restore the optimal level of investment.

The second part of the paper presents the generalized model to characterize the direction of the distortion, when a simplified but more realistic income and money distribution is reproduced. It presents a model where households face both idiosyncratic and aggregate shocks, and participate infrequently in financial markets. The model is developed to capture the self-insurance motive and to introduce limited participation in a tractable environment. It extends previous work based on periodic reinsurance (Alvarez et al., 2009, Khan and Thomas, 2015, Challe, Matheron, Ragot and Rubio-Ramirez, 2016) to introduce capital accumulation and a richer heterogeneity to reproduce the US money and income distribution. Optimal monetary policy is derived in this setup.

It is found that optimal monetary policy is countercyclical. Active monetary policy contributes to increase inflation after a negative technology shock and to decrease inflation after a positive technology shock. This policy generates an additional increase in the capital stock by 5% after a positive technology shock. The tradeoff faced by monetary policy is between improving capital accumulation and lowering risk-sharing, by increasing inequality. Optimal monetary policy raises welfare by a roughly 0.2% consumption equivalent through its ability both to partially insure households against the aggregate risk and to affect capital accumulation over the business cycle. This welfare gain is high compared to the gains of eliminating business cycles in representative agent economies. To my knowledge, this paper is the first to analyse optimal monetary policy with capital accumulation, limited participation and aggregate shocks. All these results are derived with flexible prices. This assumption is made to identify the key mechanisms. The potential new effects generated by nominal frictions are discussed as concluding remarks.

The rest of the Introduction is the literature review. Section 1 presents the simple model, where distortions of the market economy and the optimal monetary policy are identified. Section 2 presents the general model to quantify the mechanisms. Section 3 is the Conclusion.

Related literature.

Limited Participation and money demand. This paper considers limited participation as the key friction for monetary policy.[1] Limited participation is indeed a modeling strategy that is consistent with the data (Bricker, Dettling, Henriques, Hsu, Moore, Sabelhaus, Thompson and Windle, 2014 shows that roughly half of the US population participates in financial markets). The work of Alvarez and Lippi (2009) and (2014) or Ragot (2014) shows that models with limited participation in financial markets can reproduce the distribution of money. Khan and Thomas (2015) prove that this friction is useful to reproduce the correlation between consumption and the short-run interest rate. On the fiscal side, Kaplan and Violante (2014) show that limited participation can explain the high marginal propensity to consume after a fiscal shock. In this literature, the contribution of the current paper is to identify the optimal policy with capital accumulation.

Optimal monetary policy and redistribution. Monetary policy with heterogeneous agents was first studied in economies without capital (labeled pure *currency economies* by Wallace, 2014). In these models, money is the only store of value. (Bewley, 1983; Kehoe, Levine and Woodford, 1992; Algan, Challe and Ragot, 2010; Grossman and Weiss, 1983 or the search-theoretic model of Kiyotaki and Wright, 1993 as in Williamson, 2006 or Gomis-Porqueras and Sanches, 2013 among many others). Recently, Auclert (2017) identifies the redistributive channels of monetary policy with nominal asset and a fixed capital stock. Concerning optimal policies, Lippi, Ragni and Trachter (2015) derive optimal policy in a two-agent economy with aggregate shocks, following Scheinkman and Weiss (1986). Bilbiie (2008), Motta and Tirelli (2012), Nuno and Thomas (2017), Challe (2017) and Bilbiie

[1] Limited participation models were first introduced in monetary economics to rationalize the liquidity effect of money injections as the seminal contribution of Grossman and Weiss (1983) and Rotemberg (1984)

and Ragot (2017) derive optimal policy with heterogeneous agents and additional frictions in the goods or labor market. All these models abstract from capital accumulation, and optimal monetary policy is a trade-off between consumption-smoothing and insurance. Recent models in the search-theory of money consider both money and capital to characterize optimal policies at the steady state (Aruoba and Chuch, 2010; Boragan, Waller and Wright, 2011). Using limited participation as the key friction, I can derive new results with aggregate technology shocks. For instance, optimal monetary policy can be either procyclical or countercyclical depending on the persistence of the shock. Finally, the redistributive effects of monetary policy generate new results absent from representative agent models, studied in Chari and Kehoe (1999).

Old literature. It is interesting to note that both the market failure induced by monetary saving, and the role of monetary policy in affecting the incentives to save, were discussed by Hayek. The idea that monetary policy can induce capital accumulation was indeed strongly defended by Hayek (and all the Austrian school) and called ”forced saving”. (Hayek, 1967). He argued forcefully that monetary policy shouldn't generate excessive fluctuations in the investment rate. The present analysis highlights an additional constraint on monetary policy: It shouldn't increase too much inequality over the business cycle.

1. THE SIMPLE MODEL

The simple model is based on two simplifying assumptions. First, some agents always participate in financial markets, whereas others never participate. Second, non-participating households use money to smooth deterministic income fluctuations. These assumptions are relaxed in the quantitative analysis presented in Section 2.

Time is discrete and periods are indexed by $t = 0, 1...$ The model features a closed economy populated by a continuum of households indexed by i and uniformly distributed along the unit interval, as well as a representative firm. Households have a CRRA utility function $u(c) = (c^{1-\sigma} - 1)/(1-\sigma)$ if $\sigma \neq 1$ and $u(c) = \log(c)$ if $\sigma = 1$. The discount factor is β. It is assumed that the economy is composed of two types of households. There is a fraction $\Omega > 0$ of agents, denoted as $N-$households, who must pay a fixed cost κ^N each time they want to participate in financial markets. The remaining fraction $1 - \Omega$ of households, denoted as $P-$households, don't pay any cost to participate in financial markets. The cost κ^N is determined in Section 1.2 below. It is high enough that N-households never participate in financial markets. All households can participate in the money market at no cost[2].

1.1. **Agents.**

1.1.1. *Non-participating households.* $N-$households are denoted by the superscript n. A fraction $\Omega/2$ consumes in odd periods and receives labor income in even periods. The other fraction $\Omega/2$ consumes in even periods and receives labor income in odd periods, which is a modeling strategy similar to Woodford (1990). When working, households supply one unit of labor and get a nominal wage W_t. In all periods, households receive a net nominal transfer $P_t\tau_t$, where P_t is the price of one unit of final goods and τ_t is the transfer in real terms. As these households will not participate in financial markets, they use money only to smooth consumption[3].

Households cannot issue money. When they consume, it is guessed (and checked) that they spend all their money holdings, and the condition for this to be the case is provided below. From now on, real variables are denoted with lowercase. For instance, M_t^n is the nominal amount of money held by the households at the beginning of each period, and the real amount is $m_t^n = M_t^n/P_t$. Denote as c_t^n the consumption of non-participating households in period t, then $P_t c_t^n = M_{t-1}^n + P_t\tau_t$, or in real terms:

$$c_t^n = \frac{m_{t-1}^n}{1+\pi_t} + \tau_t \tag{1}$$

where $\pi_t = P_t/P_{t-1} - 1$ is the net inflation rate. When households do not consume, their money demand is their total income. As they spent all their money the previous period,

[2] This participation costs structure is a simplification of the general framework of Alvarez, Atkeson and Kehoe (2002). It allows studying limited participation in a simple environment, as in Alvarez and Lippi (2014) for instance. Introducing participation cost for participating households would only complicate the algebra.

[3] Money has a positive value because it is a store of value in this infinite-horizon setting. The theory of money embedded in the simple model is thus from Samuelson (1958).

their real money demand is:

$$m_t^n = w_t + \tau_t \tag{2}$$

From standard dynamic optimization, the condition for households not to hold money when they consume is:

$$u'\left(c_t^n\right) > \beta^2 E_t \frac{1}{1+\pi_{t+1}} \frac{1}{1+\pi_{t+2}} u'\left(c_{t+2}^n\right)$$

The condition is that marginal gain from consuming an additional unit of money in the current period is higher than the expected gain from consuming it two periods ahead.

1.1.2. *Participating Households.* Variables concerning $P-$households are indicated by the superscript p. These households supply one unit of labor every period. $P-$households can buy two types of assets: money, and the capital of firms. As money will always be a dominated asset[4], participating household never hold money in equilibrium. In period t, they buy a quantity k_{t+1}^p of financial assets, which yield a real return $1+r_{t+1}$ between period t and $t+1$. The budget constraint of a representative $P-$households is, in real terms:

$$k_{t+1}^p + c_t^p = w_t + \tau_t + \left(1+r_t\right) k_t^p, \tag{3}$$

where c_t^p is real consumption, w_t is real labor income and $\left(1+r_t\right) k_t^p$ is the return of financial savings. Standard intertemporal utility maximization yields the Euler equation:

$$u'\left(c_t^p\right) = \beta E_t \left(1+r_{t+1}\right) u'\left(c_{t+1}^p\right), \tag{4}$$

and the transversality condition is $\lim_{\tau\to\infty} \beta^\tau E u'\left(c_{t+\tau}^p\right) k_{t+\tau}^p = 0$

1.1.3. *Firms.* There is a unit mass of firms, which produce with capital and labor. Capital must be installed one period before production, and it fully depreciates in production. The production function is Cobb-Douglas with a capital share μ : $Y_t = A_t K_t^\mu L_t^{1-\mu}$ and where K_t, L_t and A_t are respectively the capital stock, the labor hired and the technology level at

[4]It will be assumed that shocks are small enough such that the zero lower bound does not bind in the equilibrium under consideration, so that money is a dominated asset.

the beginning of period t. Profit maximization is $\max_{K,L} A_t K_t^{\mu} L_t^{1-\mu} - w_t L_t - (1+r_t) K_t$. It yields the following two first-order conditions:

$$w_t = (1-\mu) A_t K_t^{\mu} L_t^{-\mu} \tag{5}$$

$$1 + r_t = \mu A_t K_t^{\mu-1} L_t^{1-\mu} \tag{6}$$

The level of technology A_t is the only exogenous stochastic process in the economy. $A_t = e^{a_t}$, where a_t follows an AR(1) process:

$$a_t = \rho^a a_{t-1} + \varepsilon_t^a \tag{7}$$

where the shock ε_t^a is a white noise $\mathcal{N}\left(0, \sigma_a^2\right)$. The steady-state technology level is defined as $A_t = 1$.

Monetary policy and taxes. The new money is created by lump-sum transfers given to all households. The central bank creates a nominal quantity of money M_t^{CB}. The real quantity is $m_t^{CB} = M_t^{CB}/P_t$. Denote as M_t^{tot} the total nominal quantity of money. The law of motion of M^{tot} is simply $M_t^{tot} = M_{t-1}^{tot} + M_t^{CB}$, or in real terms:

$$m_t^{tot} = \frac{m_{t-1}^{tot}}{1+\pi_t} + m_t^{CB} \tag{8}$$

The new money is created by a lump-sum transfer to all agents:

$$\tau_t = m_t^{CB} \tag{9}$$

1.2. **Equilibrium definition, steady state.** There are four markets in this economy. First, the equilibrium of the money market is:

$$m_t^{tot} = \frac{\Omega}{2} m_t^n \tag{10}$$

The previous equality stipulates that half of the N-households ($\Omega/2$) hold money at the end of each period. As only $P-$households participate in financial markets, the equilibrium of the financial markets is:

$$(1-\Omega) k_t^p = K_t \tag{11}$$

The goods market equilibrium is:

$$(1-\Omega) c_t^p + \frac{\Omega c_t^n}{2} + K_{t+1} = Y_t \tag{12}$$

As half $N-$households and all $P-$households supply one unit of labor, the labor market equilibrium is $L_t = L$, where:

$$L \equiv 1 - \Omega + \frac{\Omega}{2} = 1 - \Omega/2 \tag{13}$$

Given the process for the technology and for a given monetary policy, an equilibrium of this economy is a sequence of individual choices and prices $\{c_t^n, m_t^n, k_t^p, c_t^p, r_{t,}\pi_t, w_t\}$ and a sequence of money stock, central bank profits and taxes $\{m_t^{tot}, m_t^{CB}, \tau_t\}$ such that agents make optimal choices, the aggregate quantity of money is consistent with money creation and markets clear.

1.2.1. *Steady state.* The steady state of the model gives first insights. In steady state, real variables are constant and indicated with a star. For instance, $A^* = 1$. The real interest rate is given by the Euler equation of participating agents (4). It implies that $1 + r^* = 1/\beta$. One easily deduces the steady-state capital stock from equations (5) and (6): $K^* = L\left(\mu\beta\right)^{\frac{1}{1-\mu}}$ and $w^* = (1-\mu)\left(\beta\mu\right)^{\frac{\mu}{1-\mu}}$. As π^* is the steady-state net inflation rate, using equations (2)-(9), one finds the consumption of $N-$households:

$$c^{n*} = (1-\mu)\left(\beta\mu\right)^{\frac{\mu}{1-\mu}} \frac{1 + \frac{1}{\beta}\frac{\Omega}{2}\pi^*}{1 + \left(1 - \frac{1}{\beta}\frac{\Omega}{2}\right)\pi^*}$$

The consumption of $P-$households is given by:

$$c^{p*} = \left(1 + (1-\beta)\frac{L}{1-\Omega}\frac{\mu}{1-\mu} + \pi^* \frac{\Omega/2}{1 + \pi^*\left(1 - \Omega/2\right)}\right)\left(\beta\mu\right)^{\frac{\mu}{1-\mu}}$$

One can check that $C^{tot*} = (1-\Omega)\,c^{p*} + \frac{\Omega}{2}c^{n*}$ does not depend on the inflation rate, but that c^{p*} increases with π whereas c^{n*} decreases with π^*.

Indeed, only the money-holders pay the inflation tax, whereas the proceed of the inflation tax is equally given back to all households. As a consequence, steady-state inflation is just a transfer from money holders ($N-$households) to non-money holders ($P-$households). This transfer doesn't affect output because the real interest rate is pinned down by preferences of participating agents. [5]

[5] $N-$households don't participate in financial markets if the participation cost κ^N is high enough such that the total return of their saving in money would be higher than the one in financial markets. This provides a lower bound to the participation cost of N-households: $\kappa^N > (1-\mu)\left(\beta\mu\right)^{\frac{\mu}{1-\mu}}\left(\frac{1}{\beta} - \frac{1}{1+\pi^*}\right)$. It

1.3. **Discussion of the model.** Before solving the model, it may be useful to discuss the assumptions. This model is designed to be the simplest model with two features: 1) Money is used by non-participating agents to smooth consumption, as in the Baumol-Tobin literature 2) Participating agents invest in capital, affecting prices in general equilibrium. It can be thought either as a monetary extension of Woodford (1990) or as introducing capital accumulation in simplified limited participation models along the lines of Alvarez et al. (2009) or Khan and Thomas (2015).

To obtain a tractable framework, this simple model doesn't admittedly consider other margins studied in the literature. First, the participation structure is fixed because there is constant segmentation across households. This modeling strategy, followed by Alvarez et al. (2009) or Khan and Thomas (2015) can be generalized at the cost of more algebra. For instance, Alvarez et al. (2002) consider a model where the participation decision may change in the business cycle to obtain a more time-varying velocity of money. Kaplan and Violante (2014) also consider this margin in a quantitative model. Second, there is no change in price dispersion after a money shock in my model. This could be obtained either with Calvo-type sticky prices or with flexible prices and good market segmentation as in Williamson (2008). The gain of these two simplifying assumptions is to be able to introduce capital accumulation in general equilibrium and to be able to derive optimal policies in a transparent way, what is new in this literature to the best of my knowledge.

A third implication of this simple model is more problematic. The distribution of money is not very realistic, what may be misleading as the main mechanism will rely on the redistributive effects of monetary policy. This limit of the simple model will justify the introduction of a more general setup in Section 2.

1.4. **Optimal allocation and steady state comparison.** Without loss of generality, it is assumed that the planner gives a weight ω_p to $P-$households and a weight 1 to $N-$households.

is assumed that this inequality is fulfilled and that shocks are small enough such $N-$households never participate in financial markets.

The tilde is used to indicate the optimal allocation. For instance $\tilde{c}_t^n$ is the optimal consumption of a N-household in period t. The intertemporal social welfare function is:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{\Omega}{2} u\left(\tilde{c}_t^n\right) + \omega_p \left(1-\Omega\right) u\left(\tilde{c}_t^p\right) \right) \tag{14}$$

and the resource constraint of the planner is:

$$\frac{\Omega \tilde{c}_t^n}{2} + \left(1-\Omega\right) \tilde{c}_t^p + \tilde{K}_t = A_t \tilde{K}_{t-1}^{\mu} L^{1-\mu} \tag{15}$$

Solving the program one finds:

$$\tilde{c}_t^n / \tilde{c}_t^p = \omega_p^{-\frac{1}{\sigma}} \tag{16}$$

In words, the ratio of consumption of participating and non-participating households is constant over the business cycle, which is a direct implication of full risk-sharing. With this property the Euler equation is:

$$u'\left(\tilde{c}_t^p\right) = \beta E_t \left(1+\tilde{r}_{t+1}\right) u'\left(\tilde{c}_{t+1}^p\right) \tag{17}$$

where $1+\tilde{r}_t = \mu A_t \tilde{K}_t^{\mu-1} L^{1-\mu}$ is the marginal productivity of capital in the optimal allocation. The resource constraint of the planner is $\tilde{K}_{t+1} + \left(\frac{\Omega}{2} \left(\omega_p\right)^{-\frac{1}{\sigma}} + \left(1-\Omega\right) \right) \tilde{c}_t^p = A_t \tilde{K}_t^{\mu} L^{1-\mu}$. This budget constraint and the Euler equation (17) fully characterize the optimal allocation.

One can now compare the market and optimal allocation in steady state, i.e. when there is no money creation $m_t^{CB} = 0$ and where $A = 1$. The following Proposition summarizes the result. All proofs are in the Appendix.

Proposition 1. *1) The steady state value of the capital stock is optimal in the market economy:* $K^* = \tilde{K}^*$.

2)In addition, if

$$\omega_p = \left(1 + \frac{\mu}{1-\mu} \left(1-\beta\right) \frac{1-\Omega/2}{1-\Omega}\right)^{\sigma} \tag{18}$$

then the steady-state market equilibrium is optimal when $\pi^* = 0$*:* $c^{n*} = \tilde{c}^{n*}$ *and* $c^p = \tilde{c}^{p*}$.

First, in the market economy, the capital stock is optimal in steady state. Indeed, the steady-state real interest rate (and thus the marginal return on capital) is pinned down by the households' discount factor in the market and optimal economy: $1+r^* = 1+\tilde{r}^* = \frac{1}{\beta}$

(see equations 4 and 17). Second, for the value of the weight ω^p given in the Proposition, the inflation rate $\pi = 0$ generates the optimal steady-state market allocation[6]. As steady-state inflation is only a transfer across households, there exists a pareto weight such that the optimal inflation rate is 0. In what follows, it is assumed that ω^p has the value given in the Proposition and that the optimal steady-state inflation rate is thus $\pi = 0$, but it should be clear that all the results below are valid for an arbitrary weight ω^p. Considering the case where the optimal steady-state inflation rate is 0 simplifies the algebra, in order to focus on the business cycle distortions implied by limited participation. [7]

1.5. **Distortions in the market economy.** To identify the distortions of the market economy in the business cycle, the allocation is first analyzed under the assumption that monetary policy is inactive: $m_t^{CB} = 0$. As a consequence, the nominal money stock $\bar{M}^{tot}$ is constant and $\tau_t = 0$ in all periods. Using the money market equilibrium and money demand (2) and (10), one finds that the price level in each period is $P_t = \frac{2\bar{M}^{tot}}{\Omega w_t}$, and the inflation rate is thus $1 + \pi_t = w_{t-1}/w_t$. As a consequence, from (1), the consumption of non-participating households is simply:

$$c_t^n = w_t$$

The structure of the equilibrium is thus quite simple. Non-participating households consume the real wage in all periods. The model is thus close to consumer-saver models, where some agents consume all their income (as Judd, 1985 for an early example).

To save some space, the difference between the optimal and market allocation is summarized in two Propositions. The proofs are based on a first-order approximation of the model and they are left in Appendix.

Proposition 2. *If $\sigma = 1$, the market and the optimal allocations are the same.*

The first result is that the existence of distortions depends on the utility function: When households have log-utility, the market allocation is optimal (at the first order) even when

[6]The fact that the optimal steady-state inflation rate does not necessarily produce the Friedman rule in an heterogeneous-agent economy is the standard result of Kehoe et al. (1992)

[7]When markets are complete and when all households participate in financial markets, it is easy to check that the dynamics of aggregate consumption and capital are the same in the optimal allocation.

there is limited participation in financial markets. Indeed, the steady-state is optimal and, with log-utility, households consume a constant share of their income in all periods, which is independent from factor prices. Hence, this share is the same in the market and optimal economies.

Following the business-cycle literature, the case $\sigma > 1$ and σ close to 1 is considered as the relevant one (see Hall, 2010, chap. 2 for a survey). With these assumptions we can derive additional analytical results about capital accumulation.[8]

To do so, denote as $\frac{\partial \hat{\tilde{K}}}{\partial \varepsilon^a}$ the contemporaneous increase in the total capital stock after a small TFP shock in the *optimal* allocation, and as $\frac{\partial \hat{K}}{\partial \varepsilon^a}$ the contemporaneous increase in the total capital stock in the *market* allocation. We now compare $\frac{\partial \hat{\tilde{K}}}{\partial \varepsilon^a}$ and $\frac{\partial \hat{K}}{\partial \varepsilon^a}$.

Proposition 3. *If σ is close to 1, there is a threshold $\bar{\rho} > 0$, such that:*
- If $\rho < \bar{\rho}$ then $\frac{\partial \hat{\tilde{K}}}{\partial \varepsilon^a} > \frac{\partial \hat{K}}{\partial \varepsilon^a}$
- If $\rho > \bar{\rho}$ then $\frac{\partial \hat{\tilde{K}}}{\partial \varepsilon^a} < \frac{\partial \hat{K}}{\partial \varepsilon^a}$

The Proposition states that the reaction of the total capital stock can be higher or lower than the one in the first-best allocation depending on the persistence of the technology shock. Hence, there can be either over or under investment after a technology shock. Indeed, when the persistence of the technology shock is low, the central planner would like the economy to save a lot to benefit from the temporary increase in TFP. In the market economy, participating households, do not receive all the wealth generated by investment because a part of this return is given as wages to non-participating households. The wealth effect is smaller than it would be under complete markets. As a consequence, they do not save enough. When the persistence is high, the economy experiences a high wealth effect and the central planner would like households to increase consumption to benefit from the persistent increase in TFP. Again, as participating households in the market economy do not perceive the full wealth generated by the increase in TFP, they do not increase consumption enough compared to the first best, and the capital stock is too high.

[8]When σ is very high, income effects create non-realistic behavior (such as a huge fall in saving after a positive technology shock), which prevents analytical characterization.

As a short summary, the market economy under-reacts compared to the first best allocation, for both income and substitution effect. The effect of limited participation on capital accumulation is thus surprisingly complex, as it depends on both the shape of the utility function and the persistence of the technology shock. The key mechanism identified in this Proposition is that participating households don't experience the right wealth effect after a TFP shock. We will see below that optimal policy redistributes wealth across households to restore the right wealth effects. It will do so, not by affecting marginal returns, but by generating non-distorting transfers across agents.

1.6. **Optimal monetary policy.** Optimal monetary policy is now derived in the non-linear environment. We consider that the central bank creates some money in each period after observing the state of the economy. As it is shown that the optimal monetary policy implements the first best, there are no commitment issues, as the central bank has no incentives to deviate in any period.

To identify the effect of monetary policy in the non-linear environment, one can rewrite the budget constraint of participating households, using the budget constraint (3), together with the equations (9) and (11):

$$K_{t+1} + (1-\Omega)\, c_t^p = Y_t - w_t \frac{\Omega}{2} + \underbrace{(1-\Omega)\, m_t^{CB}}_{\text{money transfer}} \tag{19}$$

The previous equality shows that monetary policy acts as a lump-sum transfer to participating households (as a general equilibrium effect). Indeed, money is transferred to all households, whereas only non-participating ones hold money and pay the inflation tax. As a consequence, money creation is a transfer from non-participating to participating households, as in the steady state. The term "money transfer" thus captures the redistributive effect of money creation between the two types of agents.

One can write the budget constraint of the central planner (15) in a similar form:

$$\tilde{K}_{t+1} + (1-\Omega)\, \tilde{c}_t^p = \tilde{Y}_t - \tilde{w}_t \frac{\Omega}{2} + \underbrace{\frac{\Omega}{2}\left(\tilde{w}_t - \tilde{c}_t^n\right)}_{\text{missing saving}} \tag{20}$$

where $\tilde{Y}_t$ and $\tilde{w}_t \equiv (1-\mu)\, A_t \tilde{K}_t^{\mu} L^{-\mu}$ are respectively the level of output and the marginal productivity of labor in the optimal allocation. The time-varying difference between the income and consumption of non-participating households appears in this budget constraint. This difference is denoted the ”missing saving”, because it is the part of the income of non-participating agents that is actually invested in the optimal allocation, but not in the market economy. This difference is key for the results. Indeed, the planner would like that non-participation agents save, but it is not the case when there is no money creation, as $c_t^n = w_t$. As a consequence, if monetary policy is able to implement a transfer to participating agents, which compensates for the ”missing saving”, it may generate the right saving decision for participating households, and thus the right wage rate and consumption for non-participating agents.

The following Proposition shows that this intuition is right.

Proposition 4. *1) An active monetary policy can implement the first best allocation.*
2) The optimal money rule has the following form:

$$m_t^{CB} = H\left(\Omega, A_t, K_t\right) \tag{21}$$

where the function H is such that $H\left(0, A_t, K_t\right) = 0$ and $H\left(\Omega, 1, K^*\right) = 0$.

The second part of the Proposition shows that the money rule depends on technology and the aggregate capital stock. When all households participate in financial markets $\Omega = 0$, the incentives to save are optimal and no money is created $H = 0$, as expected. Moreover, in steady state $H\left(\Omega, 1, K^*\right) = 0$, as the steady-state allocation is optimal. The time-variation in the money created by the central bank reproduces the transfer, which corresponds to the ”missing saving” of non-participating households identified in the discussion of equation (20). As a consequence, the consumption-saving choice of participating households is optimal. The consumption of non-participating households is thus also optimal, because of the goods market equilibrium.

One can derive some intuitions for the properties of the optimal monetary policy from Proposition 3. When the persistence of the technology shock is low (close to 0) and the utility function is not too concave, the economy under-invests after a positive technology shock. Optimal monetary policy increases capital accumulation, and it is thus procyclical. When the persistence of the technology shock is high (close to 1), then the market economy accumulates too much capital after a positive technology shock. Optimal monetary policy decreases capital accumulation after a persistent positive technology shock. Optimal monetary policy is thus countercyclical.

1.7. **Monetary policy or fiscal policy ?** Monetary policy can implement the first best by inducing optimal transfers across agents. One could argue that this should be the role of fiscal policy. As capital dynamic is not optimal in the business cycle, one could think that a time-varying capital tax could implement the first best. This intuition is not correct. Indeed, the distortion appears as a non-optimal wealth effect, not as a distorted marginal return on capital. To see this, assume that in period t the central planner introduces a time-varying capital tax λ_t on interest income on period t savings (the way the inflation tax is redistributed to households is irrelevant for the proof). The Euler equation of participation agents in period t is:

$$u'\left(c_t^p\right) = \beta E_t \left(1 + \left(1 - \lambda_t\right) r_{t+1}\right) u'\left(c_{t+1}^p\right)$$

where r_{t+1} is the before-tax marginal productivity of capital. If the first best is implemented, the optimal allocation must satisfies $u'\left(\tilde{c}_t^p\right) = \beta E_t \left(1 + \left(1 - \lambda_t\right) \tilde{r}_{t+1}\right) u'\left(\tilde{c}_{t+1}^p\right)$, in all periods. But it is known from (17) that $u'\left(\tilde{c}_t^p\right) = \beta E_t \left(1 + \tilde{r}_{t+1}\right) u'\left(\tilde{c}_{t+1}^p\right)$. As a consequence, we must have $0 = \lambda_t E_t \tilde{r}_{t+1} u'\left(\tilde{c}_{t+1}^p\right)$. This implies $\lambda_t = 0$, in all periods, because we assume small shocks and $r^* u'\left(c^{p*}\right) \neq 0$. The next Proposition summarizes this result.

Proposition 5. *The first best can be achieved only if capital taxes are zero in all periods.*

Finally, it should be clear that a time-varying lump-sum transfer between participating and non-participating households equal to the "missing saving" can reproduce the first-best allocation. As monetary policy described above, such policy would restore the right wealth effects without distorting the marginal incentive to save of participating agents.

Monetary policy has nevertheless a relative advantage. Indeed, optimal monetary policy depends only on aggregate variables, and the monetary authorities have no information about the identity of who is actually participating or not. Monetary policy thus requires less information-processing than fiscal policy.

1.8. **A remark on inside money and money creation.** The result about the distortion of the market economy does not depend on money being outside money. The results would be the same if money were inside money, because the time-varying return on inside money is different from the marginal productivity of capital. This result is proved in the Online Appendix, to save some space, but the intuition is simple. In general equilibrium what is not consumed must be invested. As a consequence, all the monetary savings (be it outside or inside money) are invested. The key distortion relies on the incentives to save, and thus on the return on the money.

The fact that monetary policy can implement the first best does not rely on the assumption of lump-sum money creation, and the results are the same if money is created by open-market operations. The results are thus robust to many alternative way to create money. Only in the special case where the new money is given to non-participating agents in a lump-sum manner, does monetary policy not generate redistribution across agents (as the inflation tax is paid back to the money holders), and the first best cannot be implemented. Any other process of money creation allows implementing the first best.[9]

This simple model has shown that the time-variations in the capital stock are not optimal when there is limited participation in financial markets. It has identified a new role for monetary policy, which is to affect capital accumulation in the business cycle because of only one friction, which is limited participation in financial markets. As summarized in the introduction, this role for monetary has not been identified yet, to the best of my knowledge. This simple model is based on simplifying assumptions, some of which are relaxed in the next Section.

2. THE GENERAL MODEL

The previous simple model can be obviously generalized in many dimensions, as discussed in Section 1.3. This Section considers three specific extensions, to bring the model closer to money theory and to get the direction of the redistribution right. The simple

[9] In a previous version of this paper optimal monetary policy was also studied at the zero lower bound (ZLB) as in Adam and Billi, 2010 among others. As the ZLB is analyzed in a separate literature, it is now studied in a different paper.

TABLE 1. Summary of the US distribution of money, using SCF 2004

	Households in the income distribution	
	Bottom 50%	Top 50%
Money	3,303	12,980
Income (in $)	26,000	137,000

model has indeed three shortcomings. First, non-participating agents don't have a consumption saving choice, due to the assumption that they both consume and receive their labor income every other period. This assumption excludes the cost of inflation as a distortion to the consumption-saving choice, which is typically studied in the monetary and Bewley literature. Second, participating agents don't hold money, which is inconsistent with the data, as discussed below. Reproducing a realistic (although simplified) distribution is key to assess the redistributive effet of monetary policy. Third, money is created by lump-sum transfers, which is an obvious simplification. Introducing open-market operations is important not to give monetary policy an arbitrary advantage as a tool to provide insurance.

I first discuss the shape of the money distribution, before presenting the modeling strategy. Using the Survey of Consumer Finance (SCF), two groups of US households can be identified according to their holdings of money and financial assets. First, roughly 50% of the US population doesn't participate in the stock market either directly or indirectly. This fraction is roughly constant, and non-participating households are mostly low-income households (Bricker et al., 2014). For this reason, the US population is divided into two groups of equal size: the bottom 50% and the top 50% in the income distribution. Table 1 provides summary statistics for the two groups of households.

Households in the Top 50%, have an income roughly 5 times higher than households in the Bottom 50% (137,000 compared to 26,000). A narrow definition of money, namely M1 (checking deposits and currency) is used, so as not to over-estimate the redistributive effect of monetary policy[10]. The Survey of Consumer Finance (SCF) 2004[11] is used to obtain data on checking accounts. As the SCF does not include data on currency, I use the Kaplan et al. (2014) strategy to estimate the currency holdings: In US data, the ratio of total currency to the total checking account is 32%. I thus increase the checking account of each group of households by 32%. Households in the Top 50% hold much more money than households in the Bottom 50%, but the ratio of money over income is smaller for high-income households (9.5) than for low-income households (12.7), as found by Erosa and Ventura (2002).

This money distribution is known to be best reproduced by limited-participation and incomplete-market models (Alvarez and Lippi, 2009 and 2014). Indeed, models introducing a cash-in-advance constraint, even with increasing returns-to-scale transaction technology, cannot reproduce the observed heterogeneity in money holdings because the distribution of money is very different from the distribution of consumption expenditures (Ragot, 2014). For this reason, the following model presents a generalization of the previous simple model, introducing a more general incomplete market and limited participation structure. The bottom 50% households will not participate in financial markets and smooth consumption with money, as in the Bewley model. The top 50% will participate infrequently in financial markets, as in the Baumol-Tobin model of money demand[12].

Incomplete insurance markets and limited participation models are known to be very difficult to analyze with aggregate shocks. To my knowledge, simulation techniques do not allow to study such environments in the general case and with aggregate shocks, excluding the derivation of optimal policies. To capture the essence of limited participation

[10] A broader definition would not alter money inequality, as M1 and M2 are roughly similarly distributed in the US population (Ragot, 2014), but the quantity of money (and thus the tax base of the inflation tax) would be higher.

[11] The 2004 SCF survey is used to avoid the high house prices of the 2007 survey and the low nominal interest rate in the 2010 survey. Nevertheless, it has been checked that the distribution of money does not vary a lot between the various surveys.

[12] Infrequent participation in financial markets is also studied inVissing-Jogensen, 2002 and Alvarez and Lippi, 2009.

and market incompleteness, and to be able to define an optimal monetary policy with aggregate shocks, I introduce two simplifying assumptions. First, following Alvarez et al. (2009) and Alvarez and Lippi (2014) , I assume that the participation structure is fixed. As the income levels of participating and non-participating households are very different, this assumption can be micro-founded by a participation cost, but I abstract from this to simplify the algebra. Second and more importantly, I use methodological tools to simplify incomplete market models. This modeling strategy can be thought of as an extension of Lucas (1990), who introduces perfect insurance within families, to allow for only partial insurance as in Alvarez et al. (2009), Khan and Thomas (2015), Challe et al. (2016). It is assumed that there is perfect insurance within some groups of the population living on "islands", but that there is no insurance across islands. The key modeling strategy is to design a timing of market opening such that the model generates Euler equations for each household, which are consistent with results in the incomplete insurance market literature, but where the heterogeneity is limited to a finite number of household types. It is thus not necessary to follow a large distribution of agents as in Krussel and Smith (1998). In this setup, optimal policy with aggregate shocks can be studied[13].

2.1. **Assets and production.** There are now three types of assets in this economy. The first one is money. It can be held by all households. As before, the net inflation rate between period t and period $t+1$ is denoted $\pi_{t+1} = \frac{P_{t+1}-P_t}{P_t}$. The second one consists of claims on the capital stock. The real return between period t and period $t+1$ is denoted r_{t+1}. The third one is nominal bonds. The nominal interest rate between period t and period $t+1$ is denoted i_t. Nominal bonds are introduced to model open market operations.

It is now assumed that capital doesn't fully depreciate in production, the depreciation rate being λ. Profit maximization is $\max_{K,L} A_t K^{\mu} L^{1-\mu} - wL - (r_t+\lambda) K$, where L is the

[13]In monetary economics this assumption is used for instance by Shi, 1997 to study the decentralization of exchange, without having to keep track of the money distribution. Heathcote, Storesletten and Violante, 2014 use the same modeling strategy in a model based on Constantinides and Duffie, 1996, where idiosyncratic shocks are persistent. Heathcote and Perri, 2015 also use a similar strategy. The contribution of the current paper is to generalize this strategy to a limited participation framework. See Ragot, 2017 for a survey of this literature.

labor supply in efficient units. First-order conditions for the firm are:

$$r_t + \lambda = \mu A_t K_t^{\mu-1} L_t^{1-\mu} \text{ and } w_t = (1-\mu) A_t K_t^{\mu} L_t^{-\mu} \tag{22}$$

with $A_t = e^{a_t}$, and the process for a_t given by (7).

2.2. **Households.** All households have the same CRRA period utility function $u(.)$, and have the same discount factor β. They pay lump-sum taxes denoted as τ_t. The population is composed of $\Omega^N \equiv 50\%$ of $N-$households, who do not participate in financial markets, but who can hold money. It is composed of $\Omega^P \equiv 50\%$ of households, who are denoted as participating agents or $P-$households, and who have access to both money and financial markets.

$N-$Households $N-$households don't participate in financial markets, but they face an idiosyncratic risk. Following the literature on uninsurable risk, it is assumed that $N-$households can be either employed or unemployed. An employed household stays employed next period with a probability α (and falls into unemployment with a probability $1-\alpha$). The household receives a wage w_t. When unemployed, households stay unemployed with a probability ρ (and find work with a probability $1-\rho$). The household gets a revenue from home production $\delta^N < w_t$. In other words, the transition matrix for the labor risk is:

$$\begin{bmatrix} \alpha & 1-\alpha \\ 1-\rho & \rho \end{bmatrix}$$

As this transition matrix is not time-varying, the constant fraction of employed households among $N-$households is:

$$n = \frac{1-\rho}{2-\alpha-\rho} \tag{23}$$

and the unemployment rate is $1-n$.

Insurance structure. It is assumed that $N-$households belong to a family, which has two locations. Employed households live on an island, denoted as $E-$island, where there is full risk-sharing. All employed $N-$households in the $E-$island supply one unit of labor and earn an after-tax real wage $w_t - \tau_t$. Unemployed agents live on an island, denoted as $U-$island, where there is also full risk-sharing. They get a per capita home production

δ^N. By the law of large numbers[14] there is a mass $n\Omega^N$ of households in the $E-$island and a mass $(1-n)\,\Omega^N$ in the $U-$island. Households who lose their job (with a probability $1-\alpha$) must travel from the E to the $U-$island at the end of the period, after the consumption-saving choice has been made. Households finding a job (with a probability $1-\rho$) have to travel from the U to the $E-$island at the end of the period. In each island, the consumption-saving choice is made by a representative of the family head before knowing who will leave the island, and who maximizes the welfare of the whole family. Finally, all households traveling across islands can take their money with them. To consume, households go to the consumption island where they can anonymously exchange goods against money, before going back to E or U-island, according to their employment status. Finally, households cannot issue money.

Timing of events. The sequence of actions is the following. First, at the beginning of each period, the family head pools the resources within all islands. The beginning-of-period money-holding is m_t^{NE} in the $E-$island and m_t^{NU} in the $U-$island. Second, the technology shock is revealed and production takes place. Third, the consumption-saving choice is made and households travel to the consumption island. Fourth, households' idiosyncratic shock is revealed, and households changing employment status travel across islands, carrying their money with them.

Money flows. Denote as $\tilde{m}_{t+1}^{NE}$ the quantity of money chosen by the representative of the family head in the E-island at the end of the current period (thus before the next period pooling of resources). Similarly, $\tilde{m}_{t+1}^{NU}$ is the current end-of-period money choice in the $U-$island. A measure $(1-\alpha)\,n\Omega^N$ of households travels from island the E to the $U-$island and the remaining measure $\alpha n\Omega^N$ stays in island E. A measure $(1-\rho)\,(1-n)\,\Omega^N$ of households travels from island U to island E and the remaining measure $\rho\,(1-n)\,\Omega^N$ stays in island E. As a consequence, the *per capita* beginning-of-period quantity of money in the E island is

$$m_{t+1}^{NE} = \left(\alpha n\Omega^N \tilde{m}_{t+1}^{NE} + (1-\rho)\,(1-n)\,\Omega^N \tilde{m}_{t+1}^{NU}\right) / \left(n\Omega^N\right)$$

[14] We assume that the law of large numbers is valid when applied to a continuum of variables. This law is valid using the Feldman and Gilles, 1985 or Green, 1994 construction.

and similarly for $U-$island $m_{t+1}^{NU} = \left(\left(1-\alpha\right)n\Omega^N \tilde{m}_{t+1}^{NE} + \rho\left(1-n\right)\Omega^N \tilde{m}_{t+1}^{NU}\right) / \left(\left(1-n\right)\Omega^N\right)$. As $\left(1-n\right)/n = \left(1-\alpha\right)/\left(1-\rho\right)$, one easily finds:

$$m_{t+1}^{NE} = \alpha\tilde{m}_{t+1}^{NE} + \left(1-\alpha\right)\tilde{m}_{t+1}^{NU} \tag{24}$$

$$m_{t+1}^{NU} = \left(1-\rho\right)\tilde{m}_{t+1}^{NE} + \rho\tilde{m}_{t+1}^{NU} \tag{25}$$

Program of the family head. The representative of the family head in both islands cares about the total intertemporal welfare of the whole family. As a consequence, the program of the family heads can be written compactly as[15]:

$$\max_{\{c_t^{NE}, c_t^{NU}, \tilde{m}_{t+1}^{NE}, m_{t+1}^{NU}\}_{t\geq 0}} E_0 \sum_{t=0}^{\infty} \beta^t \left(n\Omega^N u\left(c_t^{NE}\right) + \left(1-n\right)\Omega^N u\left(c_t^{NU}\right)\right)$$

where expectations are taken for the technology shock and subject to (for $t \geq 0$):

$$c_t^{NE} + \tilde{m}_{t+1}^{NE} = \frac{m_t^{NE}}{1+\pi_t} + w_t - \tau_t \tag{26}$$

$$c_t^{NU} + \tilde{m}_{t+1}^{NU} = \frac{m_t^{NU}}{1+\pi_t} + \delta^N - \tau_t \tag{27}$$

$$\tilde{m}_{t+1}^{NE}, \tilde{m}_{t+1}^{NU} \geq 0 \tag{28}$$

$$m_0^{NE}, m_0^{NU} \text{ given} \tag{29}$$

and subject to the laws of motion (24) and (25). The constraints (26) and (27) are respectively the per capita budget constraint of households in the $E-$island and $U-$island, expressed in real terms. In each island, the resources are the per capita money holdings and either the after-tax labor income or the after-tax home production. Inequality constraints (28) stipulate that households cannot issue money. Finally, the initial conditions are given. Using Lagrange coefficients, one easily finds the two constraints:

$$u'\left(c_t^{NE}\right)) \geq \beta E\left[\alpha u'\left(c_{t+1}^{NE}\right) + \left(1-\alpha\right)u'\left(c_{t+1}^{NU}\right)\right]\frac{1}{1+\pi_{t+1}}, \tag{30}$$

$$\text{and } \tilde{m}_{t+1}^{NE} = 0 \text{ if } u'\left(c_t^{NE}\right)) > \beta E\left[\alpha u'\left(c_{t+1}^{NE}\right) + \left(1-\alpha\right)u'\left(c_{t+1}^{NU}\right)\right]\frac{1}{1+\pi_{t+1}} \tag{31}$$

[15]As usual, in such a formulation $c_t^{NE}, c_t^{NU}, \tilde{m}_{t+1}^{NE}, m_{t+1}^{NU}$ should be thought of as a function of the history of events up to period t, which is here the history of aggregate shock $z^t \equiv \{z_0, ...z_t\}$ (see Sargent, Lunqvist 2003). I skip the dependence on this history to ease the exposition.

with

$$u\left(c_t^{NU}\right) = \beta E\left[(1-\rho)\,u'\left(c_{t+1}^{NE}\right) + \rho u'\left(c_{t+1}^{NU}\right)\right]\frac{1}{1+\pi_{t+1}}, \tag{32}$$

$$\text{and } \tilde{m}_{t+1}^{NU} = 0 \text{ if } u\left(c_t^{NU}\right) > \beta E\left[(1-\rho)\,u'\left(c_{t+1}^{NE}\right) + \rho u'\left(c_{t+1}^{NU}\right)\right]\frac{1}{1+\pi_{t+1}} \tag{33}$$

As was argued above, these Euler constraints have the same expression as the ones found in full-fledged incomplete-market models. In particular, the saving decision is made comparing per capita current marginal utility and future expected marginal utilities, which differ according to the employment status, with the relevant transition probabilities. The gain of the previous assumptions is that the beginning-of-period distribution of money has only two mass points, m_t^{NE} and m_t^{NU}.

$P-$households

$P-$households face the same employment risk as $N-$households, with the transition probabilities α and ρ. These households are more productive than $N-$households, and the labor supply is equivalent to κ units of labor of $N-$households. The wage they receive when employed is thus κw_t. When unemployed they get a revenue from home production equal to $\kappa\delta^N$. The parameter κ will be calibrated to match the empirical income distribution.

In addition, these households participate infrequently in financial markets[16]. It is assumed that when they participate in period t the probability that they participate in period $t+1$ is α^f (and the probability that they do not participate is $1-\alpha^f$). When they do not participate in period t the probability that they do not participate in period $t+1$ is ρ^f (and the probability that they participate is $1-\rho^f$). Finally, participating and working are two independent stochastic processes.

The fraction of participating $P-$households is $n^A = \left(1-\rho^f\right)/\left(2-\alpha^f-\rho^f\right)$, and the fraction $1-n^A$ does not participate. In addition, and as before, the fraction of employed $P-$households is n, defined in (23), and the fraction $1-n$ is unemployed. Note that to

[16]The methodological contribution of this Section is to provide a simple formulation of the households' problem under limited participation.

make the model tractable, the first assumption is to consider the participation opportunity as a Poisson process.

In addition, to keep the model simple[17] , it is assumed that $P-$households can be in two locations or "islands". All $P-$households who are either participating in financial markets or employed are on the same island, denoted as the $PA-$island. In this island, the family head pools resources and has access to the financial portfolio of the $P-$households. The fraction of P-households on the $PA-$island is $n^{PA} = n + n^A \left(1 - n^A\right)$.

$P-$households who both do not participate in financial markets and are unemployed are located on another island, denoted the $PU-$island. In this island, there is a family head who maximizes the welfare of all $P-$households, whatever their location. The measure of $P-$households on $PU-$island is $n^{PU} = \left(1 - n^A\right)(1 - n)$. Households know at the end of each period if they are participating or if they are employed next period. They have to move across islands accordingly, and can only take their money with them. As a consequence, households only hold money in the $PU-$island.

Asset flows. The flows across islands are the following. The fraction of $P-$households leaving the $PU-$island each period is the number of households who can either participate (and were not participating the previous period) or who find a job: $n^{PU}\left(1 - \rho^f\right) + n^{PU}\rho^f(1-\rho) = n^{PU}\left(1 - \rho\rho^f\right)$. The fraction of $P-$households leaving the $PU-$island is denoted H and is thus $H \equiv 1 - \rho\rho^f$. Denote as T the fraction of $P-$households leaving the $PA-$island. Flow accounting implies, $n^{PA} = (1-T)\, n^{PA} + Hn^{PU}$, or $T = n^{PU}\left(1 - \rho\rho^f\right)/n^{PA}$. As a consequence, a measure $Tn^{PA}\Omega^P$ leaves the PA island for the $PU-$island at the end of each period. The measure $(1-T)\, n^{PA}\Omega^P$ stays in the $PA-$island.

Denote as k_t^{PA}, b_t^{PA} and m_t^{PA}, the per capita beginning-of-period capital, bonds and money, respectively, in the $PA-$island. The end-of-period values (before agents move across islands) are $\tilde{k}_{t+1}^{PA}, \tilde{b}_{t+1}^{PA}$ and $\tilde{m}_{t+1}^{PA}$. Denote as m_t^{PU}, the per capita beginning-of-period

[17]In a previous version of the paper, it was assumed that P-households could be in 4 different islands, depending on being employed or unemployed, and participating or not. In addition one could easily introduce an effort choice e in $\alpha^f(e)$ to consider time-varying participation decision. The results are similar, at the cost of a substantial increase in the number of equations.

capital money in the $PU-$island (where the only asset is money). The end-of-period values (before agents move across islands) are $\tilde{m}_{t+1}^{PU}$. Following the same reasoning as for $N-$agents, we find:

$$m_{t+1}^{PA} = (1-T)\,\tilde{m}_{t+1}^{PA} + T\tilde{m}_{t+1}^{PU} \tag{34}$$

$$m_{t+1}^{PU} = \left(1-\rho\rho^{f}\right)\tilde{m}_{t+1}^{PA} + \rho^{f}\rho\tilde{m}_{t+1}^{PU} \tag{35}$$

Finally, as bonds and claims to the capital stock do not leave the PA island, we have:

$$k_{t+1}^{PA} = \tilde{k}_{t+1}^{PA} \text{ and } b_{t+1}^{PA} = \tilde{b}_{t+1}^{PA}$$

Program of the family head. The program of the representatives of the family head can be written compactly, as:

$$\max_{\{\tilde{k}_{t+1}^{PA},\tilde{b}_{t+1}^{PA},\tilde{m}_{t+1}^{PA},\tilde{m}_{t+1}^{PU},c_t^{PA},c_t^{PU}\}_{t\geq 0}} E_0\sum_{t=0}^{\infty}\beta^t\left(n^{PA}u\left(c_t^{PA}\right) + \left(1-n^{PA}\right)u\left(c_t^{PU}\right)\right)$$

subject to:

$$c_t^{PA} + \tilde{k}_{t+1}^{PA} + \tilde{b}_{t+1}^{PA} + \tilde{m}_{t+1}^{PA} = \kappa\frac{nw_t + n^A(1-n)\,\delta^N}{n^{PA}} - \tau_t \tag{36}$$
$$+ (1+r_t)\,k_t^{PA} + \frac{1+i_{t-1}}{1+\pi_t}b_t^{PA} + \frac{m_t^{PA}}{1+\pi_t},$$

$$\tilde{m}_{t+1}^{PU} + c_t^{PU} = \kappa\delta^N - \tau_t + \frac{m_t^{PU}}{1+\pi_t} \tag{37}$$

$$\tilde{m}_{t+1}^{PA}, \tilde{m}_{t+1}^{PU} \geq 0 \tag{38}$$

$$\tilde{k}_0^{PA}, \tilde{b}_0^{PA}, \tilde{m}_0^{PA}, \tilde{m}_0^{PU} \text{ given} \tag{39}$$

and the laws of motion (34) and (35). Equation (36) is the per capita budget constraint in the PA island. Note that the per capita labor income $\kappa\frac{nw_t+n^A(1-n)\delta^N}{n^{PA}}$ takes into account the share of unemployed agents in the PA island, which is $n^A(1-n)\,\delta^N$. Equation (37) is the per capita budget constraint in the $PU-$island. Finally (38) are positive constraints on money demand. Using Lagrange coefficients, one easily finds for the $PA-$island:

$$u'\left(c_t^{PA}\right) = \beta E\left(1+r_{t+1}\right) u'\left(c_{t+1}^{PA}\right) \tag{40}$$

$$u'\left(c_t^{PA}\right) = \beta E \frac{1+i_t}{1+\pi_{t+1}} u'\left(c_{t+1}^{PA}\right) \tag{41}$$

$$u'\left(c_t^{PA}\right) \geq \beta E\left[(1-T)\, u'\left(c_{t+1}^{PA}\right) + Tu'\left(c_{t+1}^{PU}\right)\right] \frac{1}{1+\pi_{t+1}}, \tag{42}$$

$$\text{and } \tilde{m}_{t+1}^{PA} = 0 \text{ if } u'\left(c_t^{PA}\right) > \beta E\left[(1-T)\, u'\left(c_{t+1}^{PA}\right) + Tu'\left(c_{t+1}^{PU}\right)\right] \frac{1}{1+\pi_{t+1}}$$

$$u'\left(c_t^{PU}\right) \geq \beta E\left[\left(1-\rho\rho^f\right) u'\left(c_{t+1}^{PA}\right) + \rho\rho^f u'\left(c_{t+1}^{PU}\right)\right] \frac{1}{1+\pi_{t+1}}, \tag{43}$$

$$\text{and } \tilde{m}_{t+1}^{PU} = 0 \text{ if } u'\left(c_t^{PU}\right) > \beta E\left[\left(1-\rho\rho^f\right) u'\left(c_{t+1}^{PA}\right) + \rho\rho^f u'\left(c_{t+1}^{PU}\right)\right] \frac{1}{1+\pi_{t+1}} \tag{44}$$

These equations summarize households' portfolio choice with incomplete markets and limited participation. The first two equations are the choices of bonds and of claims on the capital stock. As households in the $PA-$island cannot bring their stock or bonds to other islands, there is no self-insurance motive for these two assets. These assets are priced using the marginal utilities of participating households in each period. As a consequence, the Euler equations for stock and bonds are the same as the ones of a representative agent. This, again, will simplify the structure of the equilibrium. The third equation (42) determines the money choice of agents in the $PA-$island, which takes into account the fact that money can be used by households moving to the other island, with the relevant transition probabilities. Note that when $T = 0$, participating households always participate then money would be held if its expected return is at least as high as for other financial assets. In other words, the model does not deliver any role for money except the self-insurance motive against bad idiosyncratic shocks. The last equation (43) determines the money choice of agents in the PU islands, and it can be interpreted the same way.

2.3. **Money creation, government budget and market equilibria.** For the sake of realism, the new money is created by open market operations. The central bank creates a nominal quantity of money M_t^{CB}. The real quantity is $m_t^{CB} = M_t^{CB}/P_t$ and it is used to buy a real quantity b_{t+1}^{CB} of assets by open market operation (to be consistent with the households program, $b_{t+1}^{CB} = m_t^{CB}$ denotes the quantity of bonds bought in period t). Denote as M_t^{tot}

the total nominal quantity of money. The law of motion of M_t^{tot} is simply $M_t^{tot} = M_{t-1}^{tot} + M_t^{CB}$, or in real terms:

$$m_t^{tot} = \frac{m_{t-1}^{tot}}{1+\pi_t} + m_t^{CB} \tag{45}$$

The real profits of the central bank (which bought a real quantity b_t^{CB} of public debt the previous period) are $\Gamma_t = \frac{1+i_{t-1}}{1+\pi_t} b_t^{CB}$. To keep the algebra simple, and without loss of generality, we assume that $\bar{b} = 0$ and that there is no public spending. This implies that the State gives back to households the profits of the central bank. As the population is normalized to 1, this implies that taxes are:

$$\tau_t = -\frac{1+i_{t-1}}{1+\pi_t} m_{t-1}^{CB} \tag{46}$$

The capital and bond market equilibria are:

$$\Omega^P n^{PA} b_t^{PA} + b_t^{CB} = 0, \tag{47}$$

$$\Omega^P n^{PA} k_t^{PA} = K_t, \tag{48}$$

The two previous equations state that only $P-$households hold interest-bearing assets.

The goods market equilibrium is:

$$\begin{aligned}(1-n)\,\Omega^N c_t^{NU} + n\Omega^N c_t^{NE} + n^{PU}\Omega^P c_t^{PU} + n^{PA}\Omega^P c_t^{PA} + K_{t+1} = AK_t^{\alpha} L_t^{1-\alpha} + (1-\lambda)\,K_t \\ + (1-n)\,\delta^N \left(\Omega^N + \kappa\Omega^P\right)\end{aligned} \tag{49}$$

The labor market is, in efficient units:

$$L_t = n\left(\Omega^N + \kappa\Omega^P\right) \tag{50}$$

Finally, the money market equilibrium is for $t \geq 1$:

$$m_t^{tot} = \Omega^N n \tilde{m}_t^{NE} + \Omega^N (1-n)\,\tilde{m}_t^{NU} + \Omega^P n^{PA} \tilde{m}_t^{A} + \Omega^P n^{PU} \tilde{m}_t^{PU} \tag{51}$$

2.4. **Optimal monetary policy.** We now derive the optimal monetary policy in this environment, assuming that the planner can commit to the optimal policy rule, as a first benchmark. The instrument of the central planner is the quantity of money created in each period m_t^{CB}. The planner gives a Pareto weight $\omega_n = 1$ to N- households and a weight ω_p to $P-$households (without loss of generality) . The Ramsey program for the planner is the following maximization:

$$W^{CE} = \max_{\{m_t^{CB}\}_{t=0..\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\omega^N \Omega^N \left(n^{NE} u\left(c_t^{NE}\right) + n^{NU} u\left(c_t^{NU}\right) \right) + \right. \tag{52}$$

$$\left. + \Omega^P \omega_p \left(n^{PA} u\left(c_t^{PA}\right) + n^{PU} u\left(c_t^{PU}\right) \right) \right] \tag{53}$$

subject to six Euler equations (30)-(32) and (40)-(43), the four budget constraints (26), (27), (36) and (37) the first-order conditions for the firm (22), the law of motion of the quantity of money (8), the budget of the State (46), the five market equilibria (47)-(51), subject to the law of motion of the technology shock given by (7), and given initial conditions.

Given initial capital stock K_0, an equilibrium of this economy is a set $\{c_t^{NE}, c_t^{NU}, c_t^{PA}, c_t^{PU}, m_t^{NE}, m_t^{PA}, b_{t+1}^{PA}, k_{t+1}^{PA}, K_{t+1}, r_t, w_t, m_t^{tot}, m_t^{CB}, \tau_t\}_{t=0..\infty}$ which solves the planner program given the above constraints. The steady-state economy is an economy where $A_t = 1$ and where real variables are constant.

2.5. **Equilibrium structure.** The equilibrium is constructed with a guess-and-verify strategy. Indeed, some households choose not to hold money because the return on money is too low in the equilibrium under consideration. More specifically, I make the following conjecture.

Conjecture 1: Households in U, PU island do not hold money, i.e.:

$$\tilde{m}_{t+1}^{NU} = \tilde{m}_{t+1}^{PU} = 0 \tag{54}$$

This conjecture implies that only high-income households hold money. The equilibrium conditions for this conjecture to be true are:

$$u'\left(c_t^{NU}\right) > \beta E_t \left[(1-\rho)\, u'\left(c_{t+1}^{NE}\right) + \rho u'\left(c_{t+1}^{NU}\right) \right] \frac{1}{1+\pi_{t+1}} \tag{55}$$

$$u'\left(c_t^{PU}\right) > \beta E_t \left[\left(1-\rho\rho^f\right) u'\left(c_{t+1}^{PA}\right) + \rho\rho^f u'\left(c_{t+1}^{PU}\right) \right] \frac{1}{1+\pi_{t+1}} \tag{56}$$

The conjecture will be proven at the steady state for the calibration provided below, and then it will be checked that shocks are small enough such that this conjecture is satisfied in the dynamics.

2.6. **First best.** To quantify the distortions, the first best allocation is also studied. The unconstrained planner can provide the same consumption level to N and P households. As before, we note $\tilde{x}_t$ for the value of x_t chosen by the planner. The planner now chooses the consumption of N and P households, $\tilde{c}_t^N$ and $\tilde{c}_t^P$. As before, the Pareto weight for N households is normalized to 1. The objective is thus:

$$W^{FB} = \max_{\{\tilde{c}_t^{NM}, \tilde{c}_t^{NP}, \tilde{c}_t^P, \tilde{K}_{t+1}\}_{t=0..\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\Omega^N u\left(\tilde{c}_t^N\right) + \omega_p \Omega^P u\left(\tilde{c}_t^P\right) \right] \tag{57}$$

Subject to the budget constraint:

$$\begin{aligned} &\tilde{K}_{t+1} + \Omega^N \tilde{c}_t^N + \Omega^P \tilde{c}_t^P \\ &= A_t \tilde{K}_t^\alpha L^{1-\alpha} + (1-\lambda)\tilde{K}_t + (1-n)\delta^N \left(\Omega^N + \kappa \Omega^P\right) \end{aligned}$$

with L given by (50). Solving the program, one finds as before $\tilde{c}_t^N = \omega_p^{-\frac{1}{\sigma}} \tilde{c}_t^P$ and:

$$u'\left(\tilde{c}_t^p\right) = \beta E_t \left(1 + \tilde{r}_{t+1}\right) u'\left(\tilde{c}_{t+1}^p\right) \tag{58}$$

where $1 + \tilde{r}_t$ is the period t marginal productivity of capital $1 + \tilde{r}_t = \mu A_t \tilde{K}_t^{\mu-1} L^{1-\mu} + 1 - \lambda$

2.7. **Steady state.** The next proposition presents steady-state properties to show the conditions under which the conjecture equilibrium structure is valid.

Proposition 6. *1) The steady-state capital stock and aggregate consumption are the same in the market and first-best allocation.*
2) When $\pi^* = 0$, Conjecture 1 is fulfilled, and $c^{PU*} < c^{PA*}$ and $c^{NU*} < c^{NE*}$.

Proof. The proof is simple. First, the Euler equations (40) and (58) in steady state imply $1 + \tilde{r}^* = 1 + \tilde{r}^* = \frac{1}{\beta}$. As a direct consequence, the steady-state level of capital stock is optimal in the market economy, $K^* = \tilde{K}^*$, and aggregate consumption (which can be

deduced from the goods market equilibrium) is optimal, as in the simple model.
Second, under Conjecture 54, the two Euler equations (30) and (42) imply:

$$\frac{c^{PU*}}{c^{PA*}} = \left(\left[\frac{1+\pi^*}{\beta} - 1 + T \right] \frac{1}{T} \right)^{-\frac{1}{\sigma}} \text{ and } \frac{c^{NU*}}{c^{NE*}} = \left(\left[\frac{1+\pi^*}{\beta} - \alpha \right] \frac{1}{1-\alpha} \right)^{-\frac{1}{\sigma}}.$$

Plugging these expressions in (55) and (56), one finds that the two conditions (55) and (56) are fulfilled when $1+\pi^* > \beta$ (i.e. the economy is not at the Friedman rule), what includes the case $\pi^* = 0$. □

When $1+\pi^* > \beta$ then $c^{PU*} < c^{PA*}$ and $c^{NU*} < c^{NE*}$. Indeed, in this case it is costly to self-insure using money because of inflation. Households thus rationally choose to experience a fall in consumption in case of a bad idiosyncratic shock. The central planner choosing the value π^* faces a trade-off between insurance and redistribution, and the optimal value of π^* in the Ramsey problem will depend on ω_p. As in the simple model and as a normalization, I will choose the Pareto weight ω_p such that the Ramsey problem delivers an optimal steady-state net inflation rate equal to 0, when one constrains the model to deliver a realistic amount of lack of insurance (see below).

2.8. **Calibration and results.** The period is a quarter. Preference parameters are set to standard values. The discount factor is $\beta = 0.99$ and the curvature of the utility function is $\sigma = 2$ (Hall, 2010. The production function is such that the capital share is $\mu = .36$ and the depreciation rate is $\lambda = 0.025$ (Cooley and Hansen, 1989 among others). The discount factor determines the steady-state interest rate $1 + r^* = 1/\beta$, with equation (40). This and the depreciation rate determine the steady-state capital stock and the steady-state wage rate w^* per efficient unit.

The steady-state inflation rate is assumed to be 0, $\pi^* = 0$, and Pareto weights are set accordingly (see below).

Concerning the labor market, a quarterly job-separation rate and job-finding rate is estimated using Shimer (2005) methodology. The quarterly job-separation rate is 5%, such that $\alpha = 0.95$, and the quarterly job-finding rate is 79%, such that $\rho = 0.21$. The replacement rate is calibrated to match the average money holdings of households in the Bottom

50% of the income distribution. It implies a replacement rate of 0.46, which is close to the one used by Shimer (2005). Concerning inequality in income, I take $\kappa = 4.42$ to match the ratio of the income of the Top 50% over that of the Bottom 50% (see the targets below).

Two parameters, α^f and ρ^f, concern the participation structure in financial markets. To my knowledge there is no direct estimation of the participation frequency of households in financial markets. I follow the strategy of Alvarez and Lippi (2009) which is to calibrate participation frequency to match some monetary moments of the data. First, I set $\rho^f = 0.5$ and $\alpha^f = 0.85$ to match two targets. The first one is the ratio of money over income of the top 50% households. The second one is an average fall in consumption of households transiting from employment to unemployment of 5%. This value is in line with the finding of Gruber (1997) of 7%. As a consequence, the model generates a realistic amount of uninsurable risk for money holders, which is key for welfare analysis. This calibration strategy implies that 77% of the P-households participate in financial markets each period, and the probability not to participate next period, when participating, is 15%.

The process for technology is set to standard values. The persistence of technology shock is set to $\rho^a = 0.95$ and the standard deviation is $\sigma_a = 1\%$ (Cooley and Hansen, 1989). The last parameter to be determined is the Pareto weights ω_p. I choose ω_p such that the optimal inflation in the steady-state Ramsey problem is 0. This is a normalization, as we set parameter values such that the model delivers a realistic amount of lack of self-insurance (measured by the fall in consumption when experiencing a fall in income). Solving the model numerically, one finds $\omega_p = 3.33$. Table 2 presents the parameter values.

As a summary, Table 3 presents the outcome for the four targets, which are used to calibrate the four parameters κ, ρ^f, α^f and δ^N.

The model outcome for all relevant variables is provided in the Online Appendix OA5.

Money and wealth Distribution The model matches well by construction the money (M1) distribution for both type of agents. The top 50% hold 12.7% of their annual income in money and the bottom 50% hold 9.5% of their income in money, both in the data and in the model. As a consequence, the model reproduces a relevant amount of money held by

TABLE 2. Parameter values

Population (%) and Pareto weight			Preferences and technology					
Ω^N	Ω^P	ω_p	β	σ	μ	λ	ρ^a	σ_a
50	50	3.33	.99	2	.36	.025	.95	.01
Income structure			Uninsurable risk					
κ	δ^N/w		α	ρ	α^f	ρ^f		
4.42	0.46		.95	.21	.85	.5		

TABLE 3. Calibration targets and model outcome

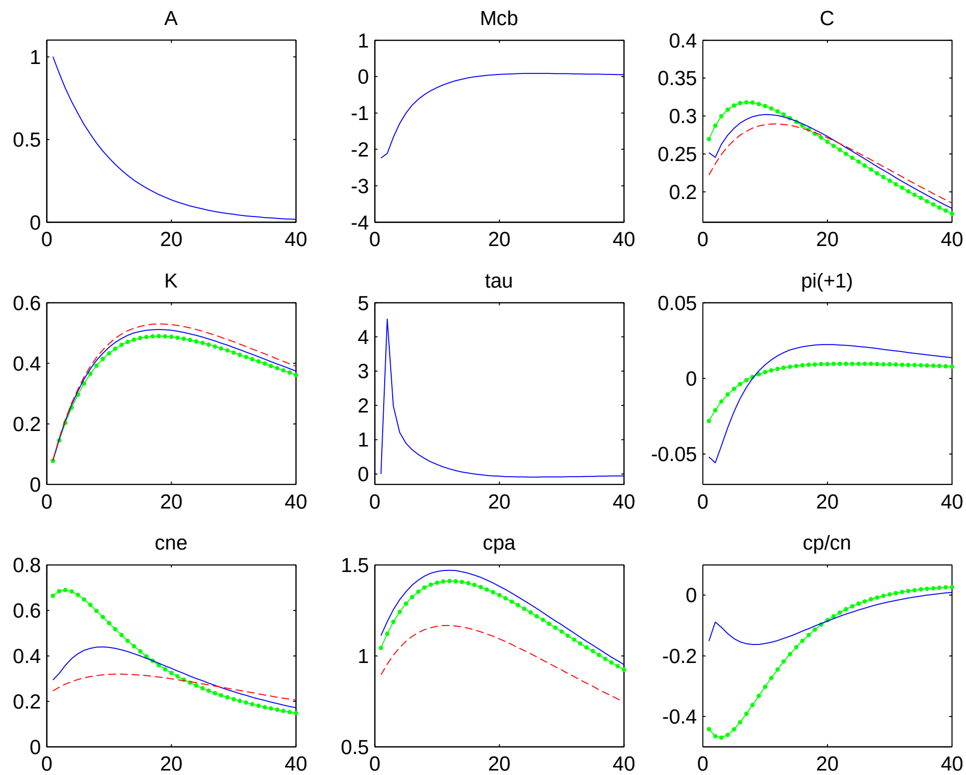
	Data	Model
Annual income T50/B50	5.3	5.3
Money/Annual Income T50	12.7	12.7
Money/Annual Income B50	9.5	9.5
Fall in consump. for unempl.	7%	5%

households. Finally, all financial wealth is held by the top 50% which is a consistent representation of the data (see Bricker et al., 2014 for the discussion of the wealth distribution using the SCF survey)

Model resolution. The gain of the assumptions made above is that the model is easy to solve numerically. In particular, to derive the optimal Ramsey policy, I consider a second-order approximation of the welfare objective and a first-order approximation of the constraints. Standard linear-quadratic methods allow deriving the optimal monetary policy of the central planner. It is then possible to compute the steady-state inflation rate generated by the solution of the Ramsey problem, as a function of the Pareto weight. I iterate over the Pareto weight until the steady-state optimal inflation rate is 0.

2.8.1. *The effect of optimal monetary policy.* To understand the trade-offs faced by monetary policy in this environment, the optimal monetary policy after a positive technology shock is now derived. Three economies are compared. The first one is the Ramsey allocation, where the planner solves the Ramsey problem (52). The second one is the first-best

FIGURE 1. Outcomes of the three economies after the same technology shock



Note: The green dotted line is the market economy with inactive monetary policy ($m^{CB} = 0$). The blue solid line is the market economy with a constrained efficient monetary policy. The red dotted line is the first-best allocation.

allocation, where the central planner is unconstrained and can implement the first best allocation. The third one is an inactive-policy allocation where monetary policy is inactive, i.e. the nominal money stock is constant. In this last economy, we impose that m_t^{CB} is 0. As the Pareto weights have been chosen such that the optimal steady-state inflation rate solving the Ramsey problem is 0, the steady-state inflation is the same in the Ramsey and the inactive-policy allocations. As a consequence, the gain of an active monetary policy is only the result of its ability to affect the business cycle and is not the outcome of a reduction in steady-state distortions.

We plot in Figure 1 the main variables in the three economies after the same technology shock.

The first panel (A), presents the technology shock A_t, as percentage deviation from the steady-state value. The second panel (Mcb) presents optimal money created m_t^{CB} when the central planner solves the Ramsey program. Optimal monetary policy is countercyclical. It reduces the money in circulation after a positive technology shock. Although aggregate consumption is increasing in the three economies (as represented in the third panel (C) of the first line), optimal monetary policy reduces the increase in consumption compared to the inactive-policy allocation. The first-best increase in consumption is even smaller. Optimal monetary policy generates an additional increase in the capital stock (compared to the inactive-policy allocation, as can be seen in the Panel (K)), although less than its level in the first-best allocation. Optimal monetary policy increases the capital stock by 5% at peak. It thus generates ”forced saving”. Panel (tau) are taxes and Panel (pi) is the expected inflation rate.

Comparing Ramsey and inactive-policy allocations, one can see that the contractionary monetary policy is a transfer from $N-$households to $P-$households. In the Ramsey allocation, the consumption of employed $N-$households (who are the non-participating money holders) is c^{NE} and is represented in the Panel (cne). This consumption is lower than the one in the inactive-policy allocation. In addition, the consumption of P-households who are participating in financial markets (c^{PA} represented in Panel (cpa)) is higher in the Ramsey allocation than in the inactive-policy allocation. These P-households save more after the technology shock in the Ramsey allocation than in the inactive-policy allocation, as can be seen from the path of the aggregate shock.

Optimal monetary policy thus increases the capital stock after such a shock, but undesirable redistributive effects limit the ability to restore the first best capital dynamics. To see this, Panel (cp/cn) plots consumption inequalities as the ratio of total consumption of participating households over total consumption of non-participating households. The average consumption of participating households is higher than that of non-participating households (who don’t hold the capital stock). As a consequence, the graph shows that consumption inequality decreases less in the Ramsey allocation than in the inactive-policy allocation. In other words, the contractionary monetary policy contributes to an increase in inequality, which is consistent with recent empirical findings (Coibion, Gorodnichenko,

TABLE 4. Second-order moments of key variables

Economies	Variables		
	K (%)	C^{tot}(%)	$corr(C^{tot}_{t-1}, C^{tot}_t)$
Inactive-policy	3.03	13.1	0.988
Ramsey	3.15	13.0	0.990
First-best	3.26	13.0	0.992

Kueng and Silvia, 2012). The change in inequality in the first best allocation is not plotted, as consumption inequality is constant in this case.

Before presenting the welfare implications, Table 4 reports the second-order moments of aggregate variables for the three allocations. The volatility of the capital stock is higher in the Ramsey allocation compared to the inactive-policy allocation, but it remains lower than in the first best allocation. This implies that the volatility of aggregate consumption falls in the Ramsey allocation compared to the inactive-policy allocation. In other words, the capital stock does not react enough (and consumption reacts too much) to the technology shock when monetary policy is inactive. In addition, the autocorrelation of aggregate consumption increases in the Ramsey allocation, compared to the inactive-policy allocation: More volatile capital stock translates into smoother aggregate consumption, because it is a way to save more in good times (and less in bad times), as can be seen from Figure 1.

2.8.2. *Welfare gains of an active monetary policy.* One can compute the welfare gains of an active monetary policy (i.e. solving the Ramsey program) compared to an inactive monetary policy, for both P and N households. I follow the standard measure of consumption equivalent in the heterogeneous-agents literature. I first compute the average welfare for the inactive allocation of $N-$households by simulating the economy with inactive policy for 10,000 periods.

I then compute the ex-ante welfare as $W^N_{IN} = \sum_{t=0}^{\infty} \beta^t \left(n^{NE} u \left(c^{NE}_{IN,t} \right) + n^{NU} u \left(c^{NU}_{IN,t} \right) \right)$ (where IN stands for the inactive-policy allocation). Similarly, I compute the ex-ante welfare of P households in the inactive allocation $W^P_{IN} = \sum_{t=0}^{\infty} \beta^t \left(n^{PA} u \left(c^{PA}_{IN,t} \right) + n^{PU} u \left(c^{PU}_{IN,t} \right) \right)$.

I then compute the ex-ante welfare for the Ramsey allocation by simulating the economy where the central planner solves the Ramsey problem for 10,000 periods. I can similarly compute the ex-ante welfare of the P and N households in the Ramsey allocation. This gives W^N_{Ramsey} and W^P_{Ramsey}. The consumption equivalent is the average increase in consumption that N and P households would need to enjoy in the inactive-policy allocation to have the same ex-ante welfare as in the Ramsey allocation. Mathematically, one computes Δ^N and Δ^P such that:

$$\sum_{t=0}^{\infty} \beta^t \left(n^{NE} u \left(c^{NE}_{IN,t} \left(1 + \Delta^N \right) \right) + n^{NU} u \left(c^{NU}_{IN,t} \left(1 + \Delta^N \right) \right) \right) = W^N_{Ramsey}$$

$$\sum_{t=0}^{\infty} \beta^t \left(n^{PA} u \left(c^{PA}_{R,t} \left(1 + \Delta^P \right) \right) + n^{PU} u \left(c^{PU}_{R,t} \left(1 + \Delta^P \right) \right) \right) = W^P_{Ramsey}$$

One finds that $\Delta^N = 0.39\%$ and $\Delta^P = 0.19\%$. Optimal monetary policy increases the welfare of both types of agents, but it increases the welfare of $N-$households more than the welfare of $P-$households. Monetary policy increases consumption smoothing (which is not optimal because some households do not participate in financial markets). This benefits relatively more $N-$households, who do not have access to financial markets. Note that the welfare gains are much higher than the gains from eliminating business cycles in representative agent economies.

3. Concluding remarks

This paper derives implications for optimal monetary policy of limited participation in financial markets, as a key friction to understand money demand. The distortions generated by this simple friction are surprisingly complex. Investment can be either too high or too low in the business cycle compared to the first best allocation. Indeed, participating agents don't face the optimal wealth effect after technology shocks. The optimal policy is not to distort the marginal return on investment, but to generate time-varying redistribution. As a consequence, the redistributive effect of monetary policy is actually a tool to improve capital accumulation.

In the more general model matching a stylized money distribution, it has been shown that the market economy underinvests after a typical technology shock, when monetary policy is inactive. In this setup, monetary policy is countercyclical : optimal money creation falls after a positive technology shock, but it has to balance a positive effect on capital accumulation and undesirable redistributive effects.

The additional interest of the general model is to present a tractable incomplete insurance market model with limited participation, which generates a simple but realistic distribution of money. The model is simple enough to perform a welfare analysis with aggregate shocks. Admittedly, this model is not a full-fledged quantitative analysis of the business cycle, as many other relevant ingredients for business cycle analysis are missing. An obvious path for future work is to introduce other frictions in this model, such as nominal frictions or a search-and-matching model of the labor market, to study their interaction with limited participation. These interactions may help to think about important trade-offs for monetary policy and concerning capital accumulation.

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Appendix A. Proof of Proposition 1

Using the equations for the optimal program (Equations 15 and 17), one finds $1+\tilde{r}^* = \frac{1}{\beta}$ and $\tilde{K}^* = L(\Omega)(\beta\mu)^{1/(1-\mu)}$, the same values as in the market economy. As a consequence, $K^* = \tilde{K}^*$ and $Y = \tilde{Y}^*$, and total consumption is the same in both economy $C^{tot*} = \tilde{C}^{tot*}$. The central planner allocation implies $\tilde{c}^{n*}/\tilde{c}^{p*} = \omega_p^{-\frac{1}{\sigma}}$. The market allocation is, when $\pi = 0$:

$$c^{p*} = \left(\mu(1-\beta)\frac{1-\Omega/2}{1-\Omega} + 1 - \mu\right)(\beta\mu)^{\frac{\mu}{1-\mu}} \text{ and } c^{n*} = w^* = (1-\mu)(\mu\beta)^{\frac{\mu}{1-\mu}}$$

As total consumption is the same in the market economy and for the optimal allocation, a necessary and sufficient condition to have $c^{n*} = \tilde{c}^{n*}$ and $c^{p*} = \tilde{c}^{p*}$ is $\tilde{c}^{p*}/\tilde{c}^{n*} = c^{p*}/c^{n*}$. Using the three previous equations, this condition can be written as the one given by equality (18).

Appendix B. Proof of Propositions 2

To prove the proposition, the solution of the linear model is compared to the solution of the linearized equations characterizing the first best allocation. The proportional deviation of the variables x_t to its steady-state value is denoted $\hat{x}_t$, that is $x_t = x^*(1+\hat{x}_t)$. Linearizing and simplifying the model (3)-(6), (11) and (13), one finds that the dynamic of the economy is a simple two-equation model, based on the two variables $\hat{c}_t^p$ and $\hat{K}_t$:

$$E_t \hat{c}_{t+1}^p - \hat{c}_t^p = \frac{\mu - 1}{\sigma} \hat{K}_{t+1} + \frac{1}{\sigma} E_t a_{t+1} \tag{59}$$

$$\hat{K}_{t+1} + (\theta(\Omega) - 1)\, \hat{c}_t^p = \theta(\Omega) \left(\mu \hat{K}_t + a_t \right) \tag{60}$$

where the coefficient $\theta(\Omega)$ stands for:

$$\theta(\Omega) \equiv \frac{1}{\beta} \left(1 + (1 - \Omega) \frac{1 - \mu}{\mu L(\Omega)} \right)$$

The coefficient $\theta(\Omega)$ is higher than 1, $\theta(\Omega) > 1$ and decreasing in Ω. It captures the fact that a part of the return on the capital stock is paid in wages to non-participating agents. Knowing the value of the capital stock, the consumption of non-participating households is simply (using the expression of the real wage):

$$\hat{c}_t^n = a_t + \mu \hat{K}_t \tag{61}$$

The linearization of the equations characterizing the first best allocation yields:

$$E_t \widehat{\tilde{c}}_{t+1}^p - \widehat{\tilde{c}}_t^p = \frac{\mu - 1}{\sigma} \widehat{\tilde{K}}_t + \frac{1}{\sigma} E a_{t+1} \tag{62}$$

$$\left(\frac{1}{\mu\beta} - 1 \right) \widehat{\tilde{c}}_t^p + \widehat{\tilde{K}}_{t+1} = \frac{1}{\mu\beta} \left(a_t + \mu \widehat{\tilde{K}}_{t+1} \right) \tag{63}$$

and the consumption of non-participating households is simply, from the optimal consumption allocation (16):

$$\widehat{\tilde{c}}_t^n = \widehat{\tilde{c}}_t^p \tag{64}$$

Comparing the market economy (59)-(60) and the optimal allocation characterized by (62)-(63), one finds that the Euler equation has the same expression in the two economies. The only difference lies in the budget constraint. In the market economy the budget constraint is modified because a part of the return of capital is given as a wage to non-participating agents. The two equations are the same when $\Omega = 0$, as $\theta(0) = 1/(\beta\mu)$, as can be expected: when all agents participate in financial markets, the market economy is optimal.

Consider the dynamic system (59)-(60). Using (60), one can substitute $\hat{c}_t^p$ in (59), to obtain a single equation in $\hat{K}_t, \hat{K}_{t+1}$ and $\hat{K}_{t+2}$. Using the method of unknown coefficients, one finds that the capital stock has the form:

$$\hat{K}_{t+1} = B\left(\sigma,\theta\left(\Omega\right)\right)\hat{K}_t + D^a\left(\sigma,\theta\left(\Omega\right),\rho^a\right)a_t \tag{65}$$

where ρ^a is the persistence of the technology shock, and where :

$$\begin{aligned} B\left(\sigma,\theta\right) &\equiv \frac{1}{2\sigma}\left(\left(1-\mu\left(1-\sigma\right)\right)\theta+\mu+\sigma-1\right) \\ &\quad -\frac{1}{2}\sqrt{\frac{1}{\sigma^2}\left(\left(1-\mu\left(1-\sigma\right)\right)\theta+\mu+\sigma-1\right)^2-4\theta\mu} \end{aligned} \tag{66}$$

$$D^a\left(\sigma,\theta,\rho\right) \equiv \frac{\theta+\frac{\rho}{\sigma}\left(\theta\left(1-\sigma\right)-1\right)}{\frac{1}{\sigma}\left(\left(1-\mu\left(1-\sigma\right)\right)\theta+\mu+\sigma-1\right)-B\left(\sigma,\theta\right)-\rho} \tag{67}$$

Comparing (59)-(60) and (62)-(63), on can observe that the optimal and market allocations are the same when $\Omega=0$ because $\theta\left(0\right)=1/\left(\mu\beta\right)$. One thus directly find the optimal low of motion of the capital stock:

$$\widehat{\tilde{K}}_{t+1} = \tilde{B}\widehat{\tilde{K}}_t + \tilde{D}^a a_t \text{ with } \tilde{B}, \tilde{D}^a > 0 \tag{68}$$

with $\tilde{B}=B\left(\sigma,\theta\left(0\right)\right)$ and $\tilde{D}^a=D^a\left(\sigma,\theta\left(0\right),\rho\right)$.

Moreover, when $\sigma=1$, whatever the value of θ (and thus of Ω), one finds $B\left(1,\theta\right)=\mu$ and $D\left(1,\theta,\rho\right)=1$. As the consequence, the dynamics of the capital stock is the same in both economies. It is then easy to show that the consumption of both P and $N-$households is the same in both economies (using the good-market equilibrium), what concludes the proof.

Appendix C. Proof of the Proposition 3

Denote as $\frac{\partial\hat{x}_t}{\partial\varepsilon^a}$ the increase in the contemporaneous proportional deviation of the variable $\hat{x}_t$ due to a marginal increase in the innovation in the TFP process.

Assume that $\sigma=1+\varepsilon$ with ε small such that a first order expansion of $B\left(1+\varepsilon,\theta\right)$ and $D^a\left(1+\varepsilon,\theta,\rho\right)$ in ε is relevant. From (66) and (67). One finds

$$B\left(\varepsilon,\theta\right)=\mu+\left(\mu-1\right)\left(\theta-1\right)\frac{1}{2}\left(1-\frac{1}{\left(\theta+\mu\right)\left(\theta-\mu\right)}\right)\varepsilon$$

$$D^a\left(\varepsilon,\theta,\rho\right)=1+\frac{\theta-1}{\theta-\rho}\left(\left(1-\mu\right)\frac{\theta}{\theta-\mu}-\rho\right)\varepsilon$$

Using (61), the dynamic of consumption can be written as $\frac{\partial \widehat{c}^n}{\partial \varepsilon^a} = 1$ and plugging the expression of $\hat{K}_{t+1}$ given by (65), in the budget constraint of $P-$households (60), one finds $\hat{c}_t^p = \frac{\theta - D^a}{\theta - 1} a_t + \frac{\theta\mu - B}{\theta - 1}\hat{K}_t$. Hence

$$\frac{\partial \widehat{c}^p}{\partial \varepsilon^a} = \frac{\theta - D^a\left(\varepsilon, \theta, \rho\right)}{\theta - 1} \tag{69}$$

As the solution of the optimal program for the $P-$households is the same as the one in the market economy for $\theta\left(0\right)$, one has from (69):

$$\frac{\partial \widehat{c}^p}{\partial \varepsilon^a} = \frac{\theta\left(\Omega\right) - D^a\left(\varepsilon, \theta\left(\Omega\right), \rho\right)}{\theta\left(\Omega\right) - 1} \text{ and } \frac{\partial \widehat{\widehat{c}}^p}{\partial \varepsilon^a} = \frac{\theta\left(0\right) - D^a\left(\varepsilon, \theta\left(0\right), \rho\right)}{\theta\left(0\right) - 1}$$

Recall that $\theta\left(\Omega\right)$ is decreasing in Ω, hence $\theta\left(\Omega\right) < \theta\left(0\right)$ for $\Omega > 0$. Moreover, from (65) and (68), one has:

$$\frac{\partial \widehat{\widehat{K}}}{\partial \varepsilon^a} = D^a\left(\varepsilon, \theta\left(0\right), \rho\right) \text{ and } \frac{\partial \hat{K}}{\partial \varepsilon^a} = D^a\left(\varepsilon, \theta\left(\Omega\right), \rho\right)$$

The proof relies on the following Lemma:

Lemma 1. *There is a $\bar{\rho} > 0$, such that*

$$\text{If } \rho < \bar{\rho},\ D^a\left(\sigma, \theta\left(\Omega\right), \rho\right) < \tilde{D}^a\left(\sigma, \theta\left(0\right), \rho\right)$$
$$\text{If } \rho > \bar{\rho},\ D^a\left(\sigma, \theta, \rho\right) > \tilde{D}^a\left(\sigma, \theta\left(0\right), \rho\right)$$

Proof of the Lemma. For small ε :

$$D^a\left(\sigma, \theta, \rho\right) = 1 + \frac{\theta - 1}{\theta - \rho}\left(\left(1 - \mu\right)\frac{\theta}{\theta - \mu} - \rho\right)\varepsilon$$

Using the previous expression, one finds:

$$\frac{1}{\varepsilon}\frac{\partial}{\partial \theta}D^a\left(\sigma, \theta, \rho\right) = \left(\frac{1 - \rho}{\theta - \rho}\left(\left(1 - \mu\right)\frac{\theta}{\theta - \mu} - \rho\right) - \left(\theta - 1\right)\left(1 - \mu\right)\frac{\mu}{\left(\theta - \mu\right)^2}\right)\frac{1}{\theta - \rho}$$

Define $F\left(\rho\right) \equiv \frac{1-\rho}{\theta-\rho}\left(\left(1 - \mu\right)\frac{\theta}{\theta - \mu} - \rho\right) - \left(\theta - 1\right)\left(1 - \mu\right)\frac{\mu}{\left(\theta - \mu\right)^2}$. From the previous expression, the sign of $\frac{\partial}{\partial \theta}D^a\left(\sigma, \theta, \rho\right)$ is the sign of $F\left(\rho\right)$.

If $\rho > \theta\frac{1-\mu}{\theta-\mu}$ then $F\left(\rho\right) < 0$.

If $\rho < (1-\mu)\frac{\theta}{\theta-\mu}$ then $F(\rho)$ is decreasing and continuous in ρ., $F(0) > 0$ and $F\left((1-\mu)\frac{\theta}{\theta-\mu}\right) < 0$. As a consequence, there is a $\bar{\rho} > 0$, such that $F(\rho) > 0$ if $\rho < \bar{\rho}$ and $F(\rho) < 0$ if $\rho > \bar{\rho}$. As a consequence

$$\frac{\partial}{\partial\theta}D^a(\sigma,\theta,\rho) > 0 \text{ if } \rho < \bar{\rho}$$
$$\frac{\partial}{\partial\theta}D^a(\sigma,\theta,\rho) < 0 \text{ if } \rho > \bar{\rho}$$

Hence, if$\rho < \bar{\rho}$, then $D^a(\sigma,\theta(\Omega),\rho) < D^a(\sigma,\theta(0),\rho)$, and the reverse when $\rho > \bar{\rho}$, what concludes the proof of the Lemma.■

To conclude the proof of the proposition, one can use the lemma to rank the impact response for $\widehat{c}^p$, $\widehat{\tilde{c}}^p$, $\widehat{\tilde{K}}$ and $\hat{K}$.

Appendix D. Proof of Proposition 4

Define as $\tilde{g}\left(A_t,\tilde{K}_t\right)$ the optimal decision rule of the central planner : $\tilde{K}_{t+1} = \tilde{g}\left(A_t,\tilde{K}_t\right)$, solving the program (15) and (17) and which is uniquely defined by standard dynamic programming argument. Assume that in the market economy the money supply follows the rule $m_t^{CB} = H(\Omega, A_t, K_t)$ where,

$$H(\Omega, A_t, K_t) \equiv \left(\left((1-\mu)\frac{1-\Omega}{1-\frac{\Omega}{2}}+\mu\right)(K^*)^\mu L^{1-\mu} - K^*\right)\left(\frac{A_t\tilde{K}_t^\mu L^{1-\mu} - \tilde{g}\left(A_t,\tilde{K}_t\right)}{(K^*)^\mu L^{1-\mu} - K^*}\right)$$
$$-\left((1-\mu)\frac{1-\Omega}{1-\frac{\Omega}{2}}+\mu\right)A_tK_t^\mu L^{1-\mu} + \tilde{g}(A_t,K_t)$$

Although this expression is complex, it is only a function of the past state variables K_t, and on the current technology shock A_t. I now shown that the first best allocation is a solution of the program of all agents in the market economy, when monetary policy follows the previous rule. As a consequence, optimal monetary policy can implement the first best[18]. The proof is done in two steps.

First, using the budget constraint of participating households (3) and (9), one finds that the budget constraint of participating households can be written as a simple system in c_t^p

[18]It has been check that the first best allocation is the only possible equilibrium in a first order approximation of the dynamics. In other words, the equilibrium is locally unique.

and K_t (plugging the expression of H and using $\tilde{c}^{p*} = c^{p*}$)

$$\left(\frac{\Omega}{2}\left(\omega_p\right)^{-\frac{1}{\sigma}} + (1-\Omega)\right) c_t^p = A_t K_t^{\mu} L^{1-\mu} - \tilde{g}\left(A_t, K_t\right) + \frac{\frac{\Omega}{2}\left(\omega_p\right)^{-\frac{1}{\sigma}} + (1-\Omega)}{1-\Omega}\left(\tilde{g}\left(A_t, K_t\right) - K_{t+1}\right)$$

Second, we can show that the optimal decision rule $K_{t+1} = \tilde{g}\left(A_t, K_t\right)$ (the one derived from the solution of the program of the central planner) is a solution to the problem of participating households. Indeed, the program of these households can be written as

$$u'\left(c_t^p\right) = \beta E_t\left(\mu A_{t+1} K_{t+1}^{\mu-1} L^{1-\mu}\right) u'\left(c_{t+1}^p\right)$$

$$K_{t+1} + \left(\frac{\Omega}{2}\left(\omega_p\right)^{-\frac{1}{\sigma}} + (1-\Omega)\right) c_t^p = A_t K_t^{\mu} L^{1-\mu} + \frac{1}{2}\frac{\Omega}{1-\Omega}\left(\omega_p\right)^{-\frac{1}{\sigma}}\left(\tilde{g}\left(A_t, K_t\right) - K_{t+1}\right)$$

One recognizes the program of the central planer (17) and the budget constraint given in Section 1.4, with an extra term at the right hand side $\frac{1}{2}\frac{\Omega}{1-\Omega}\left(\omega_p\right)^{-\frac{1}{\sigma}}\left(\tilde{g}\left(A_t, K_t\right) - K_{t+1}\right)$, which is nul when $K_{t+1} = \tilde{g}\left(A_t, K_t\right)$. As a consequence, if $K_{t+1} = \tilde{g}\left(A_t, K_t\right)$ is a solution of the central planner program, it is also a solution of the program of $P-$households in the limited-participation economy. Hence, $c_t^p = \tilde{c}_t^p$ and $K_t = \tilde{K}_t$ and $c_t^n = \tilde{c}_t^n$ by the goods market equilibrium.