Capital accumulation and the optimal quantity of money over the business cycle

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Abstract

The paper presents a general equilibrium model where agents have limited participation in financial markets and use money to smooth consumption. This framework is consistent with recent empirical findings on money demand, and it generates a new role for monetary policy: The market allocation is not efficient because only a fraction of households participate in financial markets in each period. Optimal monetary policy substantially increases welfare by changing investment decisons over the business cycle, but adverse redistributive effects of monetary policy limit the scope for an active monetary policy. New developments in the heterogeneous-agents literature are used to develop a tractable framework with aggregate shocks, where optimal monetary policy can be analyzed.

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1 Introduction

The rapid expansion of central bank balance sheets in the US, Japan and the Euro area after the 2008 crisis has rejuvinated old but deep questions: What are the real effects of money injections? Are these effects, if any, desirable? The answers to these questions obviously depend on the desire of private agents to hold the new money, and thus on the evolution of money demand.

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The understanding of money demand, and its relationship with financial frictions, has improved thanks to empirical contributions. Recent analysis shows that households' money demand is best understood when one introduces a friction that generates limited participation in financial markets. In this case, agents use money to smooth consumption between periods at which they adjust their financial portfolio. The distribution of money across households generated by this friction is much more similar to the data than the distribution generated by alternative money demand (Alvarez and Lippi 2009; Cao et al 2012; Ragot 2014). The initial idea for this foundation for money demand dates back to Baumol and Tobin's seminal contributions, and it has been developed in the limited participation literature in monetary economics.

This paper studies the positive and normative implications of money creation in a model where money demand is based on limited participation. It is shown that this friction generates a new role for monetary policy in the business cycle. It is already known that limited participation generates some relevant sort-run effects, such as the liquidity effect of money injection: An increase in the quantity of money decreases the nominal interest rate, as only a part of the population must absorb the new money created (Lucas 1990; Alvarez, Atkeson and Edmond 2009, among others). Nevertheless, this promising literature has faced some difficulties in dealing with agents' heterogeneity (see the literature review below). This has prevented the introduction of additional features which are important for understanding the business cycle, such as long-lasting heterogeneity, aggregate shock and capital accumulation. Recent developments in the heterogenous agent literature now allow deriving new results about optimal monetary policy in these environments.

Why is limited participation in financial markets important for monetary economics? Before discussing the model, it may be useful to provide some intuitions about the new mechanisms generated by limited participation. First, if agents smooth consumption with money between dates at which they participate in financial markets, for many agents the marginal remuneration of saving is the return on money (roughly the opposite of the inflation rate), the fluctuations of which are different from the fluctuations in the marginal productivity of capital. As a consequence, agents do not have the right incentives to save in the business cycle. In the paper, we indeed provide evidence of a very low correlation between the return on money and the return on financial assets. Second, money creation generates some redistribution across agents because of the heterogeneity in money holdings under limited participation. Through this channel, monetary policy can affect and hopefully improve the saving decisions.

To study these questions, this paper first presents a simple general-equilibrium model to derive formal proofs. Then, it provides a quantitative framework to study optimal monetary policy with heterogeneous agents and aggregate shocks.

Analyzing the simple model, one first finds that the distortions generated by limited participation are surprisingly not simple. In general, both capital accumulation and risk sharing across households are not optimal, as a part of the income generated by the capital stock is distributed as wages to households who do not participate in financial markets. The direction of the distortions (for instance the over or under accumulation of capital after a technology shock) crucially depends on the persistence of the technology shock, because of income and substitution effects. Lack of risk sharing is fully characterized, which is an independant result. Money creation can restore the first-best allocation by affecting capital accumulation. For instance, to increase aggregate saving, money creation induces a transfer between non-participating and participating households, which also implements optimal consumption levels for all agents. In addition, the optimal allocation cannot be implemented by a time-varying capital tax, because it would distort the intertemporal consumption smoothing. In this sense, monetary policy is a powerful tool to restore the optimal level of investment.

The second part of the paper quantifies these effects. It presents a model where households face both idiosyncratic and aggregate shocks, and participate infrequently in financial markets. In this setup, new tools are developed to capture the self-insurance motive and and to introduce limited participation in a tractable environment. The model reproduces money and income inequalities in the US quite well. Moreover, it also reproduces a positive but low correlation between the return on money and the return on financial assets.

In this setup, optimal monetary policy is countercyclical. Active monetary policy contributes to increase inflation after a negative technology shock and to decrease inflation after a positive technology shock. This policy generates an increase in the capital stock by 5% after a positive technoogy shock. Optimal monetary policy increases welfare by a roughly 0.2% consumption equivalent through its ability both to partially insure households against the aggregate risk and to affect capital accumulation in the business cycle. This welfare gain is subtantial compared to the gains of eliminating business cycles in representative agent economies. To my knowledge, this paper is the first to analyse optimal monetary policy in an incomplete market environment with capital accumulation and aggregate shocks.

All these results are derived with flexible prices. This assumption is made to identify the key mechanisms. The potential new effects generated by nominal frictions are discussed as concluding remarks. Finally, the general outcome of this model is that optimal monetary policy affects capital accumulation[1]. In an older literature review (see below), it is interesting to observe that this view

[1]It is interesting to observe that the Federal Reserve Board produces an almost systematic assessment of business investment in press releases presenting monetary policy decisions. Moreover, reading the minutes of the Fed, one can note that the prospects of business fixed investment are almost always discussed.

of monetary policy is common to both Keynes and Hayek. Modern tools used in this paper provide rigourous proof for this claim.

The rest of the Introduction is the literature review. Section 2 presents the simple model, where distortions of the market economy and the optimal monetary policy are identified. Section 3 presents the general model to quantify the mechanisms. Section 4 is the Conclusion.

1.1 Old literature: Hayek and Keynes

Both the market failure induced by monetary saving, and the role of monetary policy in affecting the incentives to save, were discussed by Hayek and Keynes. First, the idea that expansionary monetary policy induces capital accumulation was strongly defended by Hayek (and all the Austrian school):

> "The theory that an increase of money brings about an increase of capital, which has recently become very popular under the name of 'forced saving', is even older than the one we have just been considering." (Hayek, 1931).

Hayek argued forcefully that monetary policy should be neutral. This is not a claim for a totally inactive monetary policy, but for a monetary policy that does generate excessive fluctuations in the investment rate. Using modern economic tools the notion of a neutral monetary policy is easy to define: it is the constrained-optimal money creation. The model below shows that a neutral monetary policy is an active one.

Second, as Chamley (2012) and (2014) noticed, the idea that monetary policy and incomplete markets are linked can be found in Keynes (1936), who, in his chapter on "Investment Incentives", claims that saving in money generates wrong investment incentives:

> "An individual decision to save does not, in actual fact, involve the placing of any specific forward order for consumption, but merely the cancellation of a present order. For this overlooks the fact that there is always an alternative to the ownership of real capital-assets, namely the ownership of money [...]" (Keynes, 1936).

Monetary saving is identified by Keynes as a potential distortion for the incentive to invest. The current paper can be seen as a clarification of these distortions.

1.2 Related literature

Optimal monetary policy and redistribution. Money creation with heterogeneous agents was first studied in pure *currency economies*, as defined by Wallace (2014). In these models, money is the only store of value. Three types of models can be identified: the Bewley tradition (Bewley 1983 or Kehoe, Levine and Woodford 1992; Algan, Challe and Ragot 2011), the Grossman and Weiss (1986) model (as Lippi, Ragni and Trachter 2015), and the search-theoretic model in the tradition of Kiyotaki and Wright (1993). These models define the optimal monetary policy as a trade-off between consumption-smoothing and insurance, which is generated by the redistributive effect of monetary policy. In general, the Friedman rule may not be optimal (Kehoe, Levine and Woodford 1992, or Wallace 2014 for a recent contribution), contrary to the results obtained in the representative agent framework (see Chari Kehoe and McGrattan 1999 for an overview). This trade-off is at stake in my model, but the key effect relies on capital accumulation, which cannot be captured in pure currency models.

Limited Participation and money demand. Introducing capital accumulation in microfounded models of money is still an open issue (Lagos 2013 for a recent attempt). Limited participation models seem to be a modeling strategy that is consistent with the data (Bricker 2012 shows that roughly half of the US population participates in financial markets). The work of Alvarez and Lippi (2009, 2013) shows that models with limited participation in financial markets can reproduce the distribution of money. Ragot (2014) shows that the distribution of money across households can be reproduced in a limited participation model, and is very difficult to rationalize otherwise.

Limited Participation in general equilibrium. Limited participation models were first introduced to rationalize the liquidity effect of money injections (Grossman and Weiss 1983 and Rotemberg 1984). This literature had to deal with household heterogeneity. Lucas (1990) and Fuerst (1992) use a family structure: Agents within the family are separated at the beginning of the period and join the family at the end of the period to pool risk. This outcome does not allow for persistent effects of money shocks, which are shown in this paper to be crucial. Some other tools have been introduced. Alvarez, Atkeson and Edmond (2009) use an overlapping-generation structure. Alvarez and Lippi (2009) focus on partial equilibrium to derive new results on participation rules when households face a rich stochastic structure. As a consequence, the optimal allocation cannot be studied. Finally, models with limited participation in a non-monetary environment have been recently used by Kaplan and Violante (2013) to study fiscal policy. They show that such limited participation is necessary to reproduce the effect of fiscal shocks. To my knowledge the distortions and the optimal monetary policy have not been identified in these models.

2 The simple model

The simple model allows fully charaterizing the distortions. A textbook-style presentation is provided and extensions are discussed in Section 2.7.

Time is discrete and periods are indexed by $t = 0, 1...$ The model features a closed economy populated by a continuum of households indexed by i and uniformly distributed along the unit interval, as well as a representative firm. Households have a CRRA utility function $u(c) = (c^{1-\sigma} - 1)/(1-\sigma)$ if $\sigma \neq 1$ and $u(c) = \log(c)$ if $\sigma = 1$. The discount factor is β. It is assumed that the economy is composed of two types of households. There is a fraction $\Omega > 0$ of agents, denoted as $N-$households, who must pay a fixed cost κ^N each time they want to participate in financial markets. The remaining fraction $1 - \Omega$ of households, denoted as $P-$households, don't pay any cost to participate in financial markets. The cost κ^N is determined in Section 2.2 below. It is high enough that N-households never participate in financial markets. All households can participate in the money market at no cost[2].

2.1 Agents

2.1.1 Non-participating households

$N-$households are denoted by the superscript n. A fraction $\Omega/2$ consumes in odd periods and receives labor income in even periods. The other fraction $\Omega/2$ consumes in even periods and receives labor income in odd periods, which is a modeling strategy similar to Woodford (1990). When working, households supply one unit of labor and get a nominal wage W_t. In all periods, households receive a net nominal transfer $P_t\tau_t$, where P_t is the price of one unit of final goods and τ_t is the transfer in real terms. As these households will not participate in financial markets, they use money only to smooth consumption[3].

Households cannot issue money. When they consume, it is guessed (and checked) that they spend all their money holdings, and the condition for this to be the case is provided below. From now on, real variables are denoted with lowercase. For instance, M_t^n is the nominal amount of money held by the households at the beginning of each period, and the real amount is $m_t^n = M_t^n/P_t$. Denote as c_t^n the consumption of non-participating households in period t, then $P_t c_t^n = M_{t-1}^n + P_t\tau_t$, or in real

[2] This participation costs structure is a simplification of the general framework of Alvarez et al. (2002). It allows studying limited participation in a simple environment, as in Alvarez and Lippi (2014) for instance. Introducing participation for participating househods would only complicate the algebra.

[3] Money has a positive value because it is a store of value in this infinite-horizon setting. The theory of money embedded in the simple model is thus from Samuelson (1958).

terms:

$$c_t^n = \frac{m_{t-1}^n}{1+\pi_t} + \tau_t \tag{1}$$

where $\pi_t = P_t/P_{t-1} - 1$ is the net inflation rate. When households do not consume, their money demand is their total income. As they spent all their money the previous period, their real money demand is:

$$m_t^n = w_t + \tau_t \tag{2}$$

The condition for households not to hold money when they consume is:$u'(c_t^n) > \beta^2 E_t \frac{1}{1+\pi_{t+1}} \frac{1}{1+\pi_{t+2}} u'\left(c_{t+2}^n\right)$

2.1.2 Participating Households

Variables concerning $P-$households are indicated by the superscript p. These households supply one unit of labor every period. $P-$households can buy two types of assets: money, and the capital of firms. As money will always be a dominated asset[4], participating household never hold money in equilibrium. In period t, they buy a quantity k_{t+1}^p of financial assets, which yield a real return $1 + r_{t+1}$ between period t and $t+1$. The budget constraint of a representative $P-$households is, in real terms:

$$k_{t+1}^p + c_t^p = w_t + \tau_t + (1 + r_t) k_t^p, \tag{3}$$

where c_t^p is real consumption, w_t is real labor income and $(1 + r_t) k_t^p$ is the return of financial savings. Standard intertemporal utility maximization yields the Euler equation:

$$u'(c_t^p) = \beta E_t (1 + r_{t+1}) u'\left(c_{t+1}^p\right), \tag{4}$$

and the transversality condition is $\lim_{\tau\to\infty} \beta^\tau E\left[u'\left(c_{t+\tau}^p\right) k_{t+\tau}^p\right] = 0$

2.1.3 Firms

There is a unit mass of firms, which produce with capital and labor. Capital must be installed one period before production, and it fully depreciates in production. The production function is Cobb-Douglas $Y_t = A_t K_t^\mu L_t^{1-\mu}$where K_t, L_t and A_t are respectively the capital stock, the labor hired and the technology level at the beginning of period t. Profit maximization is $\max_{K,L} A_t K_t^\mu L_t^{1-\mu} - w_t L_t - (1 + r_t) K_t$. It yields the following two first-order conditions:

$$w_t = (1 - \mu) A_t K_t^\mu L_t^{-\mu} \tag{5}$$

$$1 + r_t = \mu A_t K_t^{\mu-1} L_t^{1-\mu} \tag{6}$$

[4]It will be assumed that shocks are small enough such that the zero lower bound does not bind in the equilibrium under consideration, so that money is a dominated asset.

The level of technology A_t is the only exogenous stochastic process in the economy. $A_t = e^{a_t}$, where a_t follows an AR(1) process:

$$a_t = \rho^a a_{t-1} + \varepsilon_t^a \tag{7}$$

where the shock ε_t^a is a white noise $\mathcal{N}(0, \sigma_a^2)$. The steady-state technology level is defined as $A_t = 1$.

Monetary policy and taxes

The new money is created by lump-sum transfers given to all households. The central bank creates a nominal quantity of money M_t^{CB}. The real quantity is $m_t^{CB} = M_t^{CB}/P_t$. Denote as M_t^{tot} the total nominal quantity of money. The law of motion of M^{tot} is simply $M_t^{tot} = M_{t-1}^{tot} + M_t^{CB}$, or in real terms:

$$m_t^{tot} = \frac{m_{t-1}^{tot}}{1+\pi_t} + m_t^{CB} \tag{8}$$

The new money is created by a lump-sum transfer to all agents:

$$\tau_t = m_t^{CB} \tag{9}$$

2.2 Equilibrium definition, steady state and participation cost

There are four markets in this economy. First, the equilibrium of the money market is:

$$m_t^{tot} = \frac{\Omega}{2} m_t^n \tag{10}$$

The previous equality stipulates that half of the N-households ($\Omega/2$) hold money at the end of each period. As only $P-$households participate in financial markets, the equilibrium of the financial markets is:

$$(1-\Omega)\, k_t^p = K_t \tag{11}$$

The goods market equilibrium is:

$$(1-\Omega)\, c_t^p + \frac{\Omega c_t^n}{2} + K_{t+1} = Y_t \tag{12}$$

As half $N-$households and all $P-$households supply one unit of labor, the labor market equilibrium is $L_t = L$, where:

$$L \equiv 1 - \Omega + \frac{\Omega}{2} = 1 - \Omega/2 \tag{13}$$

Given the process for the technology and for a given monetary policy, an equilibrium of this economy is a sequence of individual choices and prices $\{c_t^n, m_t^n, k_t^p, c_t^p, r_{t,}\pi_t, w_t\}$ and a sequence of money stock, central bank profits and taxes $\{m_t^{tot}, m_t^{CB}, \tau_t\}$ such that agents make optimal choices, the aggregate quantity of money is consistent with money creation and markets clear.

2.2.1 Steady state

The steady state of the model gives first insights. In steady state, real variables are constant and indicated with a star. For instance, $A^* = 1$. The real interest rate is given by the Euler equation of participating agents (4). It implies that $1 + r^* = 1/\beta$. One easily deduces the steady-state capital stock from equations (5) and (6): $K^* = L(\mu\beta)^{\frac{1}{1-\mu}}$ and $w^* = (1-\mu)(\beta\mu)^{\frac{\mu}{1-\mu}}$. As π^* is the stady-state net inflation rate, using equations (2)-(9), one finds the consumption of $N-$households:

$$c^{n*} = (1-\mu)(\beta\mu)^{\frac{\mu}{1-\mu}} \frac{1 + \frac{1}{\beta}\frac{\Omega}{2}\pi^*}{1 + \left(1 - \frac{1}{\beta}\frac{\Omega}{2}\right)\pi^*}$$

The consumption of $P-$households is given by:

$$c^{p*} = \left(1 + (1-\beta)\frac{L}{1-\Omega}\frac{\mu}{1-\mu} + \pi^* \frac{\Omega/2}{1 + \pi^*(1-\Omega/2)}\right)(\beta\mu)^{\frac{\mu}{1-\mu}}$$

One can check that $C^{tot*} = (1-\Omega)c^{p*} + \frac{\Omega}{2}c^{n*}$ does not depend on the inflation rate, but that c^{p*} increases with π whereas c^{n*} decreases with π^*. Steady-state inflation is just a transfer from money holders ($N-$households) to non-money holders ($P-$households), without affecting total output.

Participation cost. $N-$households don't participate in financial markets if the participation cost κ^N is high enough such that the total return of their saving in money would be higher than the one in financial markets: $w^*/(1+\pi^*) > (1+r^*)w^* - \kappa^N$. This provides a lower bound to the participation cost of N-households: $\kappa^N > (1-\mu)(\beta\mu)^{\frac{\mu}{1-\mu}}\left(\frac{1}{\beta} - \frac{1}{1+\pi^*}\right)$. It is assumed that this inequality is fulfilled and that shocks are small enough such $N-$households never participate in financial markets.

2.3 Optimal allocation and steady state comparison

The optimal allocation is defined as a benchmark to study the distortions of the market economy. Without loss of generality, it is assumed that the planner gives a weight ω_p to $P-$households and a weight 1 to $N-$households. The tilde is used to indicate the optimal allocation. For instance $\tilde{c}_t^n$ is the optimal consumption of a N-household in period t. The intertemporal social welfare function is:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{\Omega}{2}u(\tilde{c}_t^n) + \omega_p(1-\Omega)u(\tilde{c}_t^p)\right) \tag{14}$$

and the resource constraint of the planner is:

$$\frac{\Omega\tilde{c}_t^n}{2} + (1-\Omega)\tilde{c}_t^p + \tilde{K}_t = A_t\tilde{K}_{t-1}^{\mu}L^{1-\mu} \tag{15}$$

Solving the program one finds:

$$\tilde{c}_t^n = \omega_p^{-\frac{1}{\sigma}}\tilde{c}_t^p \tag{16}$$

In words, the ratio of consumption of participating and non-participating households is constant over the business cycle. With this property the Euler equation is:

$$u'\left(\tilde{c}_t^p\right) = \beta E_t \left(1+\tilde{r}_{t+1}\right) u'\left(\tilde{c}_{t+1}^p\right) \tag{17}$$

where $1+\tilde{r}_t = \mu A_t \tilde{K}_t^{\mu-1} L^{1-\mu}$ is the marginal productivity of capital in the optimal allocation. The resource constraint of the planner is $\tilde{K}_{t+1} + \left(\frac{\Omega}{2}\left(\omega_p\right)^{-\frac{1}{\sigma}} + (1-\Omega)\right)\tilde{c}_t^p = A_t \tilde{K}_t^{\mu} L^{1-\mu}$. This budget constraint and the Euler equation (17) fully characterize the optimal allocation.

First, one can compare the market and optimal allocation in steady state, i.e. when there is no money creation $m_t^{CB} = 0$ and where $A = 1$. The following Proposition summarizes the result. All proofs are in the Appendix.

Proposition 1 *1) The steady state value of the capital sotck is optimal in the market economy:* $K^* = \tilde{K}^*$.

2)In addition, if

$$\omega_p = \left(1 + \frac{\mu}{1-\mu}(1-\beta)\frac{1-\Omega/2}{1-\Omega}\right)^{\sigma} \tag{18}$$

then the steady-state market equilibrium is optimal when $\pi^* = 0$*:* $c^{n*} = \tilde{c}^{n*}$ *and* $c^p = \tilde{c}^{p*}$.

First, in the market economy the capital stock is optimal in steady state. Indeed, the steady-state real interest rate (and thus the marginal return on capital) is pinned down by the households' discount factor in the market and optimal economy: $1+r^* = 1+\tilde{r}^* = \frac{1}{\beta}$ (see equations 4 and 17). Second, for the value of the weight ω^p given in the Proposition, the inflation rate $\pi = 0$ generates the optimal steady-state market allocation[5]. As steady-state inflation is only a transfer across households, there exists a pareto weight such that the optimal inflation rate is 0. In what follows, it is assumed that ω^p has the value given in the Proposition and that the optimal steady-state inflation rate is thus $\pi = 0$, but it should be clear that all the results below are valid for an arbitrary weight ω^p. Considering the case where the optimal stady-state inflation rate is 0 simplifies the algebra, in order to focus on the business cycle distortions implied by limited participation.

Complete market economy. When markets are complete and when all households participate in financial markets, it is easy to check that the dynamics of aggregate consumption and capital are the same in the optimal allocation. Moreover, the ratio of consumption levels across households is constant over the business cycle, and is determined by the ratio of initial wealth, as is standard with a CRRA utility function. As a consequence, the complete-market allocation is the optimal allocation

[5] The fact that the optimal steady-state inflation rate does not necessarily produce the Friedman rule in an heterogeneous-agent economy is the standard result of Kehoe, Levine and Woodford (1992).

for a specific ratio of initial wealth. To implement the first best allocation, an initial transfer of wealth across households is thus sufficient.

2.4 Distortions in the market economy

To identify the distortions of the market economy in the business cycle, the allocation is first analyzed under the assumption that monetary policy is inactive: $m_t^{CB} = 0$. As a consequence, the nominal money stock $\bar{M}^{tot}$ is constant and $\tau_t = 0$ in all periods. Using the money market equilibrium and money demand (2) and (10), one finds that the price level in each period is $P_t = \frac{2\bar{M}^{tot}}{\Omega w_t}$, and the inflation rate is thus $1 + \pi_t = w_{t-1}/w_t$. As a consequence, from (1), the consumption of non-participating households is simply:

$$c_t^n = w_t$$

The structure of the equilibrium is thus quite simple. Non-participating households consume the real wage in all periods. The model is thus close to consumer-saver models, where some agents consume all their income (as Judd 1985, for an early example[6]).

To derive analytical proof, a first-order approximation of the model is derived. The solution of the linear model is compared to the solution of the linearized equations characterizing the first best allocation. The proportional deviation of the variables x_t to its steady-state value is denoted $\hat{x}_t$, that is $x_t = x^* (1 + \hat{x}_t)$. Linearizing the model, one finds that the dynamic of the economy is a simple two-equation model, based on the two variables $\hat{c}_t^p$ and $\hat{K}_t$:

$$E_t \hat{c}_{t+1}^p - \hat{c}_t^p = \frac{\mu - 1}{\sigma} \hat{K}_{t+1} + \frac{1}{\sigma} E_t a_{t+1} \tag{19}$$

$$\hat{K}_{t+1} + (\theta(\Omega) - 1) \hat{c}_t^p = \theta(\Omega) \left(\mu \hat{K}_t + a_t \right) \tag{20}$$

where the coefficient $\theta(\Omega)$ stands for:

$$\theta(\Omega) \equiv \frac{1}{\beta} \left(1 + (1 - \Omega) \frac{1 - \mu}{\mu L(\Omega)} \right)$$

The coefficient $\theta(\Omega)$ is higher than $1, \theta(\Omega) > 1$ and decreasing in Ω. It captures the fact that a part of the return on the capital stock is paid in wages to non-participating agents. The first equation of the system is the Euler equation of participating agents. The second equation is the linearization of the budget constraints, where the capital market equilibrium and the expression of the real interest rate r_t and of the real wage w_t as a function of the capital stock have been used.

[6] Although Judd (1985) studies a deterministic environment (and the long-run properties), he also insisted on the welfare effect of capital accumulation on agents not participating in financial markets. The current Section of the paper could be seen as a business-cycle extension of the analysis of the distortions of such economies.

Knowing the value of the capital stock, the consumption of non-participating households is simply (using the expression of the real wage):

$$\hat{c}_t^n = a_t + \mu \hat{K}_t \tag{21}$$

The linearization of the equations characterizing the first best allocation yields:

$$E_t \widehat{\widetilde{c}}_{t+1}^p - \widehat{\widetilde{c}}_t^p = \frac{\mu - 1}{\sigma} \widehat{\widetilde{K}}_t + \frac{1}{\sigma} E a_{t+1} \tag{22}$$

$$\left(\frac{1}{\mu\beta} - 1 \right) \widehat{\widetilde{c}}_t^p + \widehat{\widetilde{K}}_{t+1} = \frac{1}{\mu\beta} \left(a_t + \mu \widehat{\widetilde{K}}_{t+1} \right) \tag{23}$$

and the consumption of non-participating households is simply, from the optimal consumption allocation (16):

$$\widehat{\widetilde{c}}_t^n = \widehat{\widetilde{c}}_t^p \tag{24}$$

Comparing the market economy (19)-(20) and the optimal allocation characterized by (22)-(23), one finds that the Euler equation has the same expression in the two economies. The only difference lies in the budget constraint. In the market economy the budget constraint is modified because a part of the return of capital is given as a wage to non-participating agents. The two equations are the same when $\Omega = 0$, as $\theta(0) = 1/(\beta\mu)$, as can be expected: when all agents participate in financial markets, the market economy is optimal.

The time-varying allocations of the two economies are easy to solve. We now compare these two allocations to identify first the lack of risk sharing due to limited participation, and second the distortions in capital accumulation over the business cycle. As a summary of the behavior of the economy, we focus on the impact effect of a technology shock. Denote as $\frac{\partial \hat{x}_t}{\partial \varepsilon^a}$ the increase in the contemporaneous proportional deviation of the variable $\hat{x}_t$ due to a marginal increase in the innovation in the TFP process. We will thus focus on questions such as : Doe the capital stock increase too much or too little in the market economy after a technology shock, compared to the first best allocation?

The main results of this comparison are provided by the following three Propositions. All proofs are in the Appendix.

Proposition 2 *If $\sigma = 1$, the market and the optimal allocations are the same.*

When households have log-utility, the market allocation is optimal (at the first order) even when there is limited participation in financial markets. The reason is that, with log-utility, households consume a constant share of their income in all periods, which is independent from factor prices.

This share is the same in the market and optimal economies and does not react to the dynamics of the capital stock.

Following the business-cycle literature, the case $\sigma > 1$ and σ close to 1 is considered as the relevant one (see Hall 2010, chap. 2 for a survey). With these assumptions we can derive additional analytical results. When σ is very high, income effects create non-realistic behavior (such as a huge fall in saving after a positive technology shock), which prevents analytical characterization. The next Proposition focuses on the lack of risk-sharing in the market economy.

Proposition 3 *If σ is close to 1, there is a threshold $0 < \bar{\rho}^1 < 1$ such that:*
- If $\rho < \bar{\rho}^1$ then $0 < \frac{\partial \widehat{c}^p}{\partial \varepsilon^a} < \frac{\partial \widehat{c}^n}{\partial \varepsilon^a}$.
- If $\rho > \bar{\rho}^1$ then $\frac{\partial \widehat{c}^p}{\partial \varepsilon^a} > \frac{\partial \widehat{c}^n}{\partial \varepsilon^a} > 0$.

The Proposition presents the relative change in the consumption of the two groups of agents after a technology shock. From the equality (24), in the optimal allocation we always have $\frac{\partial \widehat{\widehat{c}}^p}{\partial \varepsilon^a} = \frac{\partial \widehat{\widehat{c}}^n}{\partial \varepsilon^a}$. In the market economy, the consumption of participating households can react more or less than the consumption of non-participating households, depending on the persistence of the technology shock, ρ. First, as the non-participating households consume their wage (which depends on the current technology shock, see equation 21), we have $\frac{\partial \widehat{c}^n}{\partial \varepsilon^a} = 1$. Second, the reaction of the consumption of participating agents depends on the dynamics of technology shock. When the technology shock is short-lived, participating households greatly increase their saving on impact to benefit from the temporary increase in technology. As a consequence, consumption does not increase a lot and $\frac{\partial \widehat{c}^p}{\partial \varepsilon^a} < \frac{\partial \widehat{c}^n}{\partial \varepsilon^a}$. When technology is very persistent, participating households experience a long-lived increase in wealth and in the return on saving. As a consequence, they consume more on impact and $\frac{\partial \widehat{c}^p}{\partial \varepsilon^a} > \frac{\partial \widehat{c}^n}{\partial \varepsilon^a}$. There is a well-defined threshold of the persistence of the technology shock for one effect to dominate the other.

Although the previous Proposition shows that risk-sharing is not optimal, it does not characterize the saving decision compared to the one in the optimal allocation. This is the subject of the next Proposition.

Proposition 4 *If σ is close to 1, there is a second threshold $\bar{\rho}^2$, with $0 < \bar{\rho}^2 < \bar{\rho}^1$, such that:*
- If $\rho < \bar{\rho}^2$ then $\frac{\partial \widehat{\widehat{K}}}{\partial \varepsilon^a} > \frac{\partial \widehat{K}}{\partial \varepsilon^a}$
- If $\rho > \bar{\rho}^2$ then $\frac{\partial \widehat{\widehat{K}}}{\partial \varepsilon^a} < \frac{\partial \widehat{K}}{\partial \varepsilon^a}$

The Proposition states that the reaction of the total capital stock can be higher or lower than the one in the first-best allocation depending, again, on the persistence of the technology shock. Hence, there can be either over or under investment after a technology shock. Indeed, when the persistence of the technology shock is low, the central planner would like the economy to save a lot to benefit from the temporary increase in TFP. In the market economy, participating households, do not receive all the returns of the transitory increase in TFP because a part of this return is given as wages to non-participating households. As a consequence, they do not save enough. When the persistence is high, the economy experiences a high wealth effect and the central planner would like households to increase consumption to benefit from the persistent increase in TFP. Again, as participating households in the market economy do not perceive the full return of the increase in TFP, they do not increase consumption enough compared to the first best, and the capital stock is too high.

As a short summary, participating households under-react in the market economy compared to the first best allocation.

2.5 Optimal monetary policy

Optimal monetary policy is now derived in the non-linear environment. We consider that the central bank creates some money in each period after observing the state of the economy. As it is shown that the optimal monetary policy implements the first best, there are no commitment issues, as the central bank has no incentives to deviate in any period.

To identify the effect of monetary policy in the non-linear environment, one can rewrite the budget constraint of participating households, using the budget constraint (3), together with the equations (9) and (11):

$$K_{t+1} + (1-\Omega)\, c_t^p = Y_t - w_t \frac{\Omega}{2} + \underbrace{(1-\Omega)\, m_t^{CB}}_{\text{money transfer}} \tag{25}$$

The previous equality shows that monetary policy acts as a lump-sum transfer to participating households (as a general equilibrium effect). One can write the budget constraint of the central planner (15) in a similar form:

$$\tilde{K}_{t+1} + (1-\Omega)\, \tilde{c}_t^p = \tilde{Y}_t - \tilde{w}_t \frac{\Omega}{2} + \underbrace{\frac{\Omega}{2}\left(\tilde{w}_t - \tilde{c}_t^n\right)}_{\text{missing saving}} \tag{26}$$

where $\tilde{Y}_t$ and $\tilde{w}_t \equiv (1-\mu)\, A_t \tilde{K}_t^{\mu} L^{-\mu}$ are respectively the optimal level of output and the marginal productivity of labor in the optimal allocation. In the previous constraint, the time-varying difference between the income and consumption of non-participating households appears as a transfer in this budget constraint. This difference is denoted the "missing saving", because it is the part of the income of non-participating agents that is actually invested, in the optimal allocation. As a consequence, if monetary policy is able to implement a transfer to participating agents, which compensates for the "missing saving", it may generate the right saving decision for participating households. The following Proposition shows that this intuition is right.

Proposition 5 *1) An active monetary policy can implement the first best allocation.*
2) The optimal money rule has the following form:

$$m_t^{CB} = H\left(\Omega, A_t, K_t\right) \tag{27}$$

where the function H is such that $H\left(0, A_t, K_t\right) = 0$ and $H\left(\Omega, 1, K^*\right) = 0$.

The second part of the Proposition shows that the money rule depends on technology and the aggregate capital stock. When all households participate in financial markets $\Omega = 0$, the incentives to save are optimal and no money is created $H = 0$, as expected. Moreover, in steady state $H\left(\Omega, 1, K^*\right) = 0$, as the steady-state allocation is optimal. The time-variation in the money created by the central bank reproduces the transfer, which corresponds to the "missing saving" of non-participating households identified in the discussion of equation (26). As a consequence, the consumption-saving choice of participating households is optimal. The consumption of non-participating households is thus also optimal, because of the goods market equilibrium.

One can derive some intuitions for the properties of the optimal monetary policy from Proposition 4. When the persistence of the technology shock is low (close to 0) and the utility function is not too concave, the economy under-invests after a positive technology shock. Optimal monetary policy increases capital accumulation, and it is thus procyclical. When the persistence of the technology shock is high (close to 1), then the market economy accumulates too much capital after a positive technology shock. Optimal monetary policy decreases capital accumulation after a persistent positive technology shock. Optimal monetary policy is thus countercyclical.

2.6 Monetary policy or fiscal policy ?

Monetary policy can implement the first best by inducing optimal transfers across agents. One could argue that this should be the role of fiscal policy. As capital dynamic is not optimal in the business cycle, one could think that a time-varying capital tax could implement the first best. This intuition is not correct. Indeed, the distortion appears as a non-optimal wealth effect, not as a distorted marginal return on capital. To see this, assume that in period t the central planner introduces a time-varying capital tax λ_t on interest income on period t savings (the way the inflation tax is redistributed to households is irrelevant for the proof). The Euler equation of participation agents in period t is:

$$u'\left(c_t^p\right) = \beta E_t \left(1 + \left(1 - \lambda_t\right) r_{t+1}\right) u'\left(c_{t+1}^p\right)$$

where r_{t+1} is the before-tax marginal productivity of capital. If the first best is implemented, the optimal allocation must satisfies $u'\left(\tilde{c}_t^p\right) = \beta E_t\left(1 + \left(1 - \lambda_t\right)\tilde{r}_{t+1}\right)u'\left(\tilde{c}_{t+1}^p\right)$, in all periods. But it is known from (17) that $u'\left(\tilde{c}_t^p\right) = \beta E_t\left(1 + \tilde{r}_{t+1}\right)u'\left(\tilde{c}_{t+1}^p\right)$. As a consequence, we must have $0 = \lambda_t E_t \tilde{r}_{t+1} u'\left(\tilde{c}_{t+1}^p\right)$. This implies $\lambda_t = 0$, in all periods, because we assume small shocks and $r^* u'\left(c^{p*}\right) \neq 0$. The next Proposition summarizes this result.

Proposition 6 *The first best can be achieved only if capital taxes are zero in all periods.*

This proposition could be seen as a Judd-Chamley type of result in a time-varying environment. In particular, Judd (1985) shows that capital taxes should be zero in steady state even if workers do not save.

Finally, it should be clear that a time-varying lump-sum transfer between participating and non-participating households equal to the "missing saving" can reproduce the first-best allocation. Monetary policy has nevertheless a relative advantage. Indeed, optimal monetary policy depends only on aggregate variables, and the monetary authorities have no information about the identity of who is actually participating or not. Monetary policy thus requires less information-processing than fiscal policy.

2.7 A remark on inside money and money creation

The result about the distortion of the market economy does not depend on money being outside money. The results would be the same if money were inside money, because the time variations of the return on inside money are different than for the marginal productivity of capital. This result is proved in the Online Appendix, to save some space, but the intuition is simple. In general equilibrium what is not consumed must be invested. As a consequence, all the monetary savings (be it outside or inside money) are invested. The key distortion relies on the incentives to save, and thus on the return on the money.

The fact that monetary policy can implement the first best does not rely on the assumption of lump-sum money creation. Only when the new money is given to non-participating agents in a lump-sum manner, does monetary policy not generate redistribution across agents (as the inflation tax is paid back to the money holders), and the first best cannot be implemented. This should be considered as a special case, and any other process of money creation allows implementing the first best.[7]

[7] In a previous version of this paper optimal monetary policy was also studied at the zero lower bound (ZLB). As the ZLB is analyzed in a separate literature, it is now studied in a different paper.

3 The general model

In this Section, a quantitative model is introduced to study optimal monetary policy with a more realistic money distribution than the one in the previous simple model. Indeed, using the Survey of Consumer Finance (SCF), two groups of US households can be identified according to their holdings of money and financial assets. First, roughly 50% of the US population doesn't participate in the stock market either directly or indirectly. This fraction is roughly constant, and non-participating households are mostly low-income households (Bricker et al., 2014). For this reason, the US population is divided into two groups of equal size: the bottom 50% and the top 50% in the income distribution. The next Table provides summary statistics for the two groups of households.

	Households in the income distribution	
	Bottom 50%	Top 50%
Money	3,303	12,980
Income (in $)	26,000	137,000

Table 1: Summary of the US distribution of money, using SCF 2004.

Households in the Top 50%, have an income roughly 5 times higher than households in the Bottom 50% (137,000 compared to 26,000). A narrow definition of money, namely M1 (checking deposits and currency) is used, so as not to over-estimate the redistributive effect of monetary policy[8]. The Survey of Consumer Finance (SCF) 2004[9] is used to obtain data on checking accounts. As the SCF does not include data on currency, I use the Kaplan and Violante (2014) strategy to estimate the currency holdings: In US data, the ratio of total currency to the total checking account is 32%. I thus increase the checking account of each group of households by 32%. Households in the Top 50% hold much more money than households in the Bottom 50%, but the ratio of money over income is smaller for high-income households (9.5) than for low-income households (12.7), as found by Erosa and Ventura (2002).

This money distribution is known to be best reproduced by limited-participation and incomplete-market models (Alvarez and Lippi, 2009, 2013). Indeed, models introducing a cash-in-advance constraint, even with increasing returns-to-scale transaction technology, cannot reproduce the observed heterogeneity in money holdings because the distribution of money is very different from the dis-

[8] A broader definition would not alter money inequality, as M1 and M2 are roughly similarly distributed in the US population (Ragot, 2014), but the quantity of money (and thus the tax base of the inflation tax) would be higher.

[9] The 2004 SCF survey is used to avoid the high house prices of the 2007 survey and the low nominal interest rate in the 2010 survey. Nevertheless, it has been checked that the distribution of money does not vary a lot between the various surveys.

tribution of consumption expenditures (Ragot 2014). For this reason, the following model presents a generalization of the previous simple model, introducing a more general incomplete market and limited participation structure. The bottom 50% households will not participate in financial markets and smooth consumption, as in the Bewley model. The top 50% will participate infrequently in financial markets, as in the Baumol-Tobin model of money demand[10].

Incomplete insurance markets and limited participation models are known to be very difficult to analyze with aggregate shocks. To my knowledge, simulation techniques do not allow to study such environments in the general case and with aggregate shocks[11]. To capture the essence of limited participation and market incompleteness, and to be able to define an optimal monetary policy with aggregate shocks, I develop methodological tools to simplify incomplete market models. The modeling strategy can be thought of as an extension of Lucas (1990), who introduces perfect insurance within families. It is assumed that there is perfect insurance within some groups of the population living on "islands", but that there is no insurance across islands. The key modeling strategy is to design a timing of market opening such that the model generates Euler equations for each household, which are consistent with results in the incomplete insurance market literature, but where the heterogeneity is limited to a finite number of household types. It is thus not necessary to follow a continuous distribution of agents as in Krusell and Smith (1998). In this setup, optimal policy with aggregate shocks can be studied[12].

3.1 Assets and production

There are now three types of assets in this economy. The first one is money. It can be held by all households. As before, the net inflation rate between period t and period $t+1$ is denoted $\pi_{t+1} = \frac{P_{t+1}-P_t}{P_t}$. The second one consists of claims on the capital stock. The real return between period t and period $t+1$ is denoted r_{t+1}. The third one is nominal bonds. The nominal interest rate between period t and period $t+1$ is denoted i_t. Nominal bonds are introduced to model open market operations.

It is now assumed that capital doesn't fully depreciate in production, the depreciation rate being

[10] Infrequent participation in financial markets is also studied in Vissing-Jorgensen (2002) and Alvarez et al. (2009).

[11] Kaplan and Violante (2014) and Ragot (2014) study this environment without aggregate shocks.

[12] In monetary economics this assumption is used for instance by Shi (1997) to study the decentralization of exchange, without having to keep track of the money distribution. Heathcote, Storesletten and Violante (2014) use the same modeling strategy in a model based on Constantinides and Duffie (1996), where idiosyncratic shocks are persistent. Heathcote and Perri (2015) also use a similar strategy. The contribution of the the current paper is to generalize this strategy to a limited participation framework.

λ. Profit maximization is $\max_{K,L} A_t K^{\mu} L^{1-\mu} - wL - (r_t + \lambda) K$, where L is the labor supply in efficient units. First-order conditions for the firm are:

$$r_t + \lambda = \mu A_t K_t^{\mu-1} L_t^{1-\mu} \text{ and } w_t = (1-\mu) A_t K_t^{\mu} L_t^{-\mu} \tag{28}$$

with $A_t = e^{a_t}$, and the process for a_t given by (7).

3.2 Households

All households have the same CRRA period utility function u, and have the same discount factor β. They pay lump-sum taxes denoted as τ_t. The population is composed of $\Omega^N \equiv 50\%$ of $N-$households, who do not participate in financial markets, but who can hold money. It is composed of $\Omega^P \equiv 50\%$ of households, who are denoted as participating agents or $P-$households, and who have access to both money and financial markets.

3.2.1 $N-$Households

$N-$households do not participate in financial markets, but they face an idiosyncratic risk. Following the literature on uninsurable risk, it is assumed that $N-$households can be either employed or unemployed. An employed household stays employed next period with a probability α (and falls into unemployment with a probability $1-\alpha$). The household receives a wage w_t. When unemployed, households stay unemployed with a probability ρ (and find work with a probability $1-\rho$). The household gets a revenue from home production $\delta^N < w_t$. In other words, the transition matrix for the labor risk is:

$$\begin{bmatrix} \alpha & 1-\alpha \\ 1-\rho & \rho \end{bmatrix}$$

As this transition matrix is not time-varying, the constant fraction of employed households among $N-$households is:

$$n = \frac{1-\rho}{2-\alpha-\rho} \tag{29}$$

and the unemployment rate is $1-n$.

Insurance structure. It is assumed that $N-$households belong to a family, which has two locations. Employed households live on an island, denoted as $E-$island, where there is full risk-sharing. All employed $N-$households in the $E-$island supply one unit of labor and earn an after-tax real wage $w_t - \tau_t$. Unemployed agents live on an island, denoted as $U-$island, where there is also full risk-sharing. They get a per capita home production δ^N. By the law of large numbers[13] there is a mass $n\Omega^N$ of households in the $E-$island and a mass $(1-n)\,\Omega^N$ in the $U-$island. Households who lose their job (with a probability $1-\alpha$) must travel from the E to the $U-$island at the end of the period, after the consumption-saving choice has been made. Households finding a job (with a probability $1-\rho$) have to travel from the U to the $E-$island at the end of the period. In each island, the consumption-saving choice is made by a representative of the family head before knowing who will leave the island, and who maximizes the welfare of the whole family. Finally, all households traveling across islands can take their money with them. To consume, households go to the consumption island where they can anonymously exchange goods against money, before going back to E or U-island, according to their employment status. Finally, households cannot issue money.

Timing of events. The sequence of actions is the following. First, at the beginning of each period, the family head pools the resources within all islands. The beginning-of-period money-holding is m_t^{NE} in the $E-$island and m_t^{NU} in the $U-$island. Second, the technology shock is revealed and production takes place. Third, the consumption-saving choice is made and households travel to the consumption island. Fourth, households' idiosyncratic shock is revealed, and households changing employment status travel across islands, carrying their money with them.

Money flows. Denote as $\tilde{m}_{t+1}^{NE}$ the quantity of money chosen by the representative of the family head in the E-island at the end of the current period (thus before the next period pooling of resources). Similarly, $\tilde{m}_{t+1}^{NU}$ is the current end-of-period money choice in the $U-$island. A measure $(1-\alpha)\,n\Omega^N$ of households travel from island the E to the $U-$island and the remaining measure $\alpha n\Omega^N$ stay in island E. A measure $(1-\rho)\,(1-n)\,\Omega^N$ of households travel from island U to island E and the remaining measure $\rho\,(1-n)\,\Omega^N$ stays in island E. As a consequence, the *per capita* beginning-of-period quantity of money in the E island is $m_{t+1}^{NE} = \left(\alpha n\Omega^N \tilde{m}_{t+1}^{NE} + (1-\rho)\,(1-n)\,\Omega^N \tilde{m}_{t+1}^{NU}\right) / \left(n\Omega^N\right)$, and similarly for $U-$island $m_{t+1}^{NU} = \left((1-\alpha)\,n\Omega^N \tilde{m}_{t+1}^{NE} + \rho\,(1-n)\,\Omega^N \tilde{m}_{t+1}^{NU}\right) / \left((1-n)\,\Omega^N\right)$. As $(1-n)/n = (1-\alpha)/(1-\rho)$, one easily finds:

$$m_{t+1}^{NE} = \alpha \tilde{m}_{t+1}^{NE} + (1-\alpha)\,\tilde{m}_{t+1}^{NU} \tag{30}$$

$$m_{t+1}^{NU} = (1-\rho)\,\tilde{m}_{t+1}^{NE} + \rho \tilde{m}_{t+1}^{NU} \tag{31}$$

Program of the family head. The representative of the family head in both islands cares about the total intertemporal welfare of the whole family. As a consequence, the program of the family

[13] We assume that the law of large numbers is valid when applied to a continuum of variables. This law is valid using the Feldman and Gilles (1958) or Green (1994) construction.

heads can be written compactly as[14]:

$$\max_{\{c_t^{NE},c_t^{NU},\tilde{m}_{t+1}^{NE},m_{t+1}^{NU}\}_{t\geq 0}} E_0\sum_{t=0}^{\infty}\beta^t\left(n\Omega^N u\left(c_t^{NE}\right)+(1-n)\,\Omega^N u\left(c_t^{NU}\right)\right)$$

where expectations are taken for the technology shock and subject to (for $t \geq 0$):

$$c_t^{NE}+\tilde{m}_{t+1}^{NE}=\frac{m_t^{NE}}{1+\pi_t}+w_t-\tau_t \tag{32}$$

$$c_t^{NU}+\tilde{m}_{t+1}^{NU}=\frac{m_t^{NU}}{1+\pi_t}+\delta^N-\tau_t \tag{33}$$

$$\tilde{m}_{t+1}^{NE},\tilde{m}_{t+1}^{NU}\geq 0 \tag{34}$$

$$m_0^{NE},m_0^{NU}\text{ given} \tag{35}$$

and subject to the laws of motion (30) and (31). The constraints (32) and (33) are respectively the per capita budget constraint of households in the $E-$island and $U-$island, expressed in real terms. In each island, the resources are the per capita money holdings and either the after-tax labor income or the after-tax home production. Inequality constraints (34) stipulate that households cannot issue money. Finally, the initial conditions are given. Using Lagrange coefficients, one easily finds the two constraints:

$$u'\left(c_t^{NE}\right))\geq\beta E\left[\alpha u'\left(c_{t+1}^{NE}\right)+(1-\alpha)\,u'\left(c_{t+1}^{NU}\right)\right]\frac{1}{1+\pi_{t+1}}, \tag{36}$$

$$\text{and }\tilde{m}_{t+1}^{NE}=0\text{ if }u'\left(c_t^{NE}\right))>\beta E\left[\alpha u'\left(c_{t+1}^{NE}\right)+(1-\alpha)\,u'\left(c_{t+1}^{NU}\right)\right]\frac{1}{1+\pi_{t+1}} \tag{37}$$

(38)

$$u\left(c_t^{NU}\right)=\beta E\left[(1-\rho)\,u'\left(c_{t+1}^{NE}\right)+\rho u'\left(c_{t+1}^{NU}\right)\right]\frac{1}{1+\pi_{t+1}}, \tag{39}$$

$$\text{and }\tilde{m}_{t+1}^{NU}=0\text{ if }u\left(c_t^{NU}\right)>\beta E\left[(1-\rho)\,u'\left(c_{t+1}^{NE}\right)+\rho u'\left(c_{t+1}^{NU}\right)\right]\frac{1}{1+\pi_{t+1}} \tag{40}$$

(41)

As was argued above, these two Euler constraints have the same expression as the ones found in full-fledged incomplete-market models. In particular, the saving decision is made comparing per capita current marginal utility and future expected marginal utilities, which differ according to the employment status, with the relevant transition probabilities. The gain of the previous assumptions is that the beginning-of-period distribution of money has only two mass points, m_t^{NE} and m_t^{NU}.

[14] As usual, in such a formulation $c_t^{NE},c_t^{NU},\tilde{m}_{t+1}^{NE},m_{t+1}^{NU}$ should be thought of as a function of the history of events up to period t, which is here the history of aggregate shock $z^t\equiv\{z_0,...z_t\}$ (see Sargent, Lunqvist 2003). I skip the dependence on this history to ease the exposition.

3.2.2 $P-$households

$P-$households face the same employment risk as $N-$households, with the transition probabilities α and ρ. These households are more productive than $N-$households, and the labor supply is equivalent to κ units of labor of $N-$households. The wage they receive when employed is thus κw_t. When unemployed they get a revenue from home production equal to $\kappa\delta^N$. The parameter κ will be calibrated to match the empirical income distribution.

In addition, these households participate infrequently in financial markets[15]. It is assumed that when they participate in period t the probability that they participate in period $t+1$ is α^f (and the probability that they do not participate is $1-\alpha^f$). When they do not participate in period t the probability that they do not participate in period $t+1$ is ρ^f (and the probability that they participate is $1-\rho^f$).

The fraction of participating $P-$households is $n^A = \left(1-\rho^f\right)/\left(2-\alpha^f-\rho^f\right)$, and the fraction $1-n^A$ does not participate. In addition, and as before, the fraction of employed $P-$households is n, defined in (29), and the fraction $1-n$ is unemployed. Note that to make the model tractable, the first assumption is to consider the participation opportunity as a Poisson process.

In addition, to keep the model simple[16], it is assumed that $P-$households can be in two locations or "islands". All $P-$households who are either participating in financial markets or employed are on the same island, denoted as the $PA-$island. In this island, the family head pools resources and has access to the financial portfolio of the $P-$households. The fraction of P-households on the $PA-$island is $n^{PA} = n + n^A\left(1-n^A\right)$.

$P-$households who both do not participate in financial markets and are unemployed are located on another island, denoted the $PU-$island. In this island, there is a family head who maximizes the welfare of all members of all $P-$households, whatever their location. The measure of $P-$households on $PU-$island is $n^{PU} = \left(1-n^A\right)(1-n)$. Households know at the end of each period if they are participating or if they are employed next period. They have to move across islands accordingly, and can only take their money with them. As a consequence, households only hold money in the $PU-$island.

Asset flows. The flows across islands are the following. The fraction of $P-$households leaving the $PU-$island each period is the number of households who can either participate (and were not

[15] The methodological contribution of this Section is to provide a simple recursive formulation of the households' problem under limited participation.

[16] In a previous version of the paper, it was assumed that P-households could be in 4 different islands, depending on being employed or unemployed, and participating or not. The results are similar, at the cost of a substancial increase in the number of equations.

participating the previous period) or who find a job: $n^{PU}\left(1-\rho^{f}\right)+n^{PU}\rho^{f}\left(1-\rho\right)=n^{PU}\left(1-\rho\rho^{f}\right)$. The fraction of $P-$households leaving the PU island is denoted H and is thus $H\equiv 1-\rho\rho^{f}$. Denote as T the fraction of $P-$households leaving the PA island. Flow accounting implies, $n^{PA}=\left(1-T\right)n^{PA}+Hn^{PU}$, or $T=n^{PU}\left(1-\rho\rho^{f}\right)/n^{PA}$. As a consequence, a measure $Tn^{PA}\Omega^{P}$ leaves the PA island for the PU island at the end of each period. The measure $\left(1-T\right)n^{PA}\Omega^{P}$ stays in the $PA-$island.

Denote as k_{t}^{PA},b_{t}^{PA} and m_{t}^{PA}, the per capita beginning-of-period capital, bonds and money, respectively, in the $PA-$island. The end-of-period values (before agents move across islands) are $\tilde{k}_{t+1}^{PA},\tilde{b}_{t+1}^{PA}$ and $\tilde{m}_{t+1}^{PA}$. Denote as m_{t}^{PU}, the per capita beginning-of-period capital money in the PU island (where the only asset is money). The end-of-period values (before agents move across islands) are $\tilde{m}_{t+1}^{PU}$. Following the same reasoning as for $N-$agents, we find:

$$m_{t+1}^{PA}=\left(1-T\right)\tilde{m}_{t+1}^{PA}+T\tilde{m}_{t+1}^{PU} \tag{42}$$

$$m_{t+1}^{PU}=\left(1-\rho\rho^{f}\right)\tilde{m}_{t+1}^{PA}+\rho^{f}\rho\tilde{m}_{t+1}^{PU} \tag{43}$$

Finally, as bonds and claims to the capital stock do not leave the PA island, we have:

$$k_{t+1}^{PA}=\tilde{k}_{t+1}^{PA}\text{ and }b_{t+1}^{PA}=\tilde{b}_{t+1}^{PA}.$$

Program of the family head. The program of the representatives of the family head can be written compactly, as:

$$\max_{\{\tilde{k}_{t+1}^{PA},\tilde{b}_{t+1}^{PA},\tilde{m}_{t+1}^{PA},\tilde{m}_{t+1}^{PU},c_{t}^{PA},c_{t}^{PU}\}_{t\geq 0}} E_{0}\sum_{t=0}^{\infty}\beta^{t}\left(n^{PA}u\left(c_{t}^{PA}\right)+\left(1-n^{PA}\right)u\left(c_{t}^{PU}\right)\right)$$

subject to:

$$c_{t}^{PA}+\tilde{k}_{t+1}^{PA}+\tilde{b}_{t+1}^{PA}+\tilde{m}_{t+1}^{PA}=\kappa\frac{nw_{t}+n^{A}\left(1-n\right)\delta^{N}}{n^{PA}}-\tau_{t} \tag{44}$$

$$+\left(1+r_{t}\right)k_{t}^{PA}+\frac{1+i_{t-1}}{1+\pi_{t}}b_{t}^{PA}+\frac{m_{t}^{PA}}{1+\pi_{t}},$$

$$\tilde{m}_{t+1}^{PU}+c_{t}^{PU}=\kappa\delta^{N}-\tau_{t}+\frac{m_{t}^{PU}}{1+\pi_{t}} \tag{45}$$

$$\tilde{m}_{t+1}^{PA},\tilde{m}_{t+1}^{PU}\geq 0 \tag{46}$$

$$\tilde{k}_{0}^{PA},\tilde{b}_{0}^{PA},\tilde{m}_{0}^{PA},\tilde{m}_{0}^{PU}\text{ given} \tag{47}$$

and the laws of motion (42) and (43). Equation (44) is the per capita budget constraint in the PA island. Note that the per capita labor income $\kappa\frac{nw_{t}+n^{A}(1-n)\delta^{N}}{n^{PA}}$ takes into account the share of unemployed agents in the PA island, which is $n^{A}\left(1-n\right)\delta^{N}$. Equation (45) is the per capita budget constraint in the PU island. Finally (46) are positive constraints on money demand. Using Lagrange coefficients, one easily finds for the PA island:

$$u'\left(c_t^{PA}\right) = \beta E\left(1+r_{t+1}\right)u'\left(c_{t+1}^{PA}\right) \tag{48}$$

$$u'\left(c_t^{PA}\right) = \beta E\frac{1+i_t}{1+\pi_{t+1}}u'\left(c_{t+1}^{PA}\right) \tag{49}$$

$$u'\left(c_t^{PA}\right) \geq \beta E\left[\left(1-T\right)u'\left(c_{t+1}^{PA}\right)+Tu'\left(c_{t+1}^{PU}\right)\right]\frac{1}{1+\pi_{t+1}}, \tag{50}$$

$$\text{and } \tilde{m}_{t+1}^{PA} = 0 \text{ if } u'\left(c_t^{PA}\right) > \beta E\left[\left(1-T\right)u'\left(c_{t+1}^{PA}\right)+Tu'\left(c_{t+1}^{PU}\right)\right]\frac{1}{1+\pi_{t+1}}$$

$$u'\left(c_t^{PU}\right) \geq \beta E\left[\left(1-\rho\rho^f\right)u'\left(c_{t+1}^{PA}\right)+\rho\rho^f u'\left(c_{t+1}^{PU}\right)\right]\frac{1}{1+\pi_{t+1}}, \tag{51}$$

$$\text{and } \tilde{m}_{t+1}^{PU} = 0 \text{ if } u'\left(c_t^{PU}\right) > \beta E\left[\left(1-\rho\rho^f\right)u'\left(c_{t+1}^{PA}\right)+\rho\rho^f u'\left(c_{t+1}^{PU}\right)\right]\frac{1}{1+\pi_{t+1}} \tag{52}$$

These equations summarize households, portfolio choice with incomplete markets and limited participation. The first two equations are the choices of bonds and of claims on the capital stock. As households in the $PA-$island cannot bring their stock or bonds to other islands, there is no self-insurance motive for these two assets. These assets are priced using the marginal utilities of participating households in each period. As a consequence, the Euler equations for stock and bonds are the same as the ones of a representative agent. This, again, will simplify the structure of the equilibrium. The third equation (50) determines the money choice of agents in the $PA-$island, which takes into account the fact that money can be used by households moving to the other island, with the relevant transition probabilities. Note that when $T = 0$, participating households always participate then money would be held if its expected return is at least as high as for other financial assets. In other words, the model does not deliver any liquidity effect for money except the self-insurance motive against bad idiosyncratic shocks. The last equation (51) determines the money choice of agents in the PU islands, and it can be interpreted the same way.

3.3 Money creation, government budget and market equilibria

For the sake of realism, the new money is created by open market operations. The central bank creates a nominal quantity of money M_t^{CB}. The real quantity is $m_t^{CB} = M_t^{CB}/P_t$ and it is used to buy a real quantity b_{t+1}^{CB} of assets by open market operation (to be consistent with the households program, $b_{t+1}^{CB} = m_t^{CB}$ denotes the quantity of bonds bought in period t). Denote as M_t^{tot} the total nominal quantity of money. The law of motion of M_t^{tot} is simply $M_t^{tot} = M_{t-1}^{tot} + M_t^{CB}$, or in real terms:

$$m_t^{tot} = \frac{m_{t-1}^{tot}}{1+\pi_t} + m_t^{CB} \tag{53}$$

The real profits of the central bank (which bought a real quantity b_t^{CB} of public debt the previous period) are $\Gamma_t = \frac{1+i_{t-1}}{1+\pi_t} b_t^{CB}$. To keep the algebra simple, and without loss of generality, we assume that $\bar{b} = 0$ and that there is no public spending. This implies that the State gives back to households the profits of the central bank. As the population is normalized to 1, this implies that taxes are:

$$\tau_t = -\frac{1+i_{t-1}}{1+\pi_t} m_{t-1}^{CB} \tag{54}$$

The capital and bond market equilibria are:

$$\Omega^P n^{PA} b_t^{PA} + b_t^{CB} = 0, \tag{55}$$

$$\Omega^P n^{PA} k_t^{PA} = K_t, \tag{56}$$

The two previous equations state that only $P-$households hold interest-bearing assets.

The goods market equilibrium is:

$$\begin{aligned}(1-n)\,\Omega^N c_t^{NU} + n\Omega^N c_t^{NE} + n^{PU}\Omega^P c_t^{PU} + n^{PA}\Omega^P c_t^{PA} + K_{t+1} = AK_t^{\alpha} L_t^{1-\alpha} \\ + (1-\lambda)\,K_t + (1-n)\,\delta^N \left(\Omega^N + \kappa\Omega^P\right)\end{aligned} \tag{57}$$

The labor market is, in efficient units:

$$L_t = n\left(\Omega^N + \kappa\Omega^P\right) \tag{58}$$

Finally, the money market equilibrium is for $t \geq 1$:

$$m_t^{tot} = \Omega^N n \tilde{m}_t^{NE} + \Omega^N (1-n)\,\tilde{m}_t^{NU} + \Omega^P n^{PA} \tilde{m}_t^{A} + \Omega^P n^{PU} \tilde{m}_t^{PU} \tag{59}$$

3.4 Optimal monetary policy

We now derive the optimal monetary policy in this environment, assuming that the planner can commit to the optimal policy rule, as a first benchmark. The instrument of the central planner is the quantity of money created in each period m_t^{CB}. The planner gives a Pareto weight $\omega_n = 1$ to N-households and a weight ω_p to $P-$households (without loss of generality) . The Ramsey program for the planner is the following maximization:

$$W^{CE} = \max_{\{m_t^{CB}\}_{t=0..\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\omega^N \Omega^N \left(n^{NE} u\left(c_t^{NE}\right) + n^{NU} u\left(c_t^{NU}\right)\right) + \right. \tag{60}$$

$$\left. + \Omega^P \omega_p \left(n^{PA} u\left(c_t^{PA}\right) + n^{PU} u\left(c_t^{PU}\right)\right)\right] \tag{61}$$

subject to six Euler equations (36)-(39) and (48)-(51), the four budget constraints (32), (33), (44) and (45) the first-order conditions for the firm (28), the law of motion of the quantity of money (8), the budget of the State (54), the five market equilibria (55)-(59), subject to the law of motion of the technology shock given by (7), and given initial conditions.

Given initial capital stock K_0, an equilibrium of this economy is a set $\{c_t^{NE}, c_t^{NU}, c_t^{PA}, c_t^{PU}, m_t^{NE}, m_t^{PA}, b_{t+1}^{PA}, k_{t+1}^{PA}, K_{t+1}, r_t, w_t, m_t^{tot}, m_t^{CB}, \tau_t\}_{t=0..\infty}$ which solves the planner program given the above constraints. The steady-state economy is an economy where $A_t = 1$ and where real variables are constant.

3.5 Equilibrium structure

The equilibrium is constructed with a guess-and-verify strategy. Indeed, some households choose not to hold money because the return on money is too low in the equilibrium under consideration. More specifically, I make the following conjecture.

Conjecture 1: Households in U, PU island do not hold money, i.e.:

$$\tilde{m}_{t+1}^{NU} = \tilde{m}_{t+1}^{PU} = 0 \tag{62}$$

This conjecture implies that only high-income households hold money. The equilibrium conditions for this conjecture to be true are:

$$u'\left(c_t^{NU}\right) > \beta E_t\left[\left(1-\rho\right)u'\left(c_{t+1}^{NE}\right) + \rho u'\left(c_{t+1}^{NU}\right)\right]\frac{1}{1+\pi_{t+1}} \tag{63}$$

$$u'\left(c_t^{PU}\right) > \beta E_t\left[\left(1-\rho\rho^f\right)u'\left(c_{t+1}^{PA}\right) + \rho\rho^f u'\left(c_{t+1}^{PU}\right)\right]\frac{1}{1+\pi_{t+1}} \tag{64}$$

The conjecture will be proven at the steady state for the calibration provided below, and then it will be checked that shocks are small enough such that this conjecture is satisfied in the dynamics.

3.6 First best

To quantify the distortions, the first best allocation is also studied. The unconstrained planner can provide the same consumption level to N and P households. As before, we note $\tilde{x}_t$ for the value of x_t chosen by the planner. The planner now chooses the consumption of N and P households, $\tilde{c}_t^N$ and $\tilde{c}_t^P$. As before, the Pareto weight for N households is normalized to 1. The objective is thus:

$$W^{FB} = \max_{\{\tilde{c}_t^{NM}, \tilde{c}_t^{NP}, \tilde{c}_t^{P}, \tilde{K}_{t+1}\}_{t=0..\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\Omega^N u\left(\tilde{c}_t^N\right) + \omega_p \Omega^P u\left(\tilde{c}_t^P\right)\right] \tag{65}$$

Subject to the budget constraint:

$$\tilde{K}_{t+1} + \Omega^N \tilde{c}_t^N + \Omega^P \tilde{c}_t^P$$
$$= A_t \tilde{K}_t^{\alpha} L^{1-\alpha} + (1-\lambda)\tilde{K}_t + (1-n)\delta^N \left(\Omega^N + \kappa \Omega^P\right)$$

with L given by (58). Solving the program, one finds as before $\tilde{c}_t^N = \omega_p^{-\frac{1}{\sigma}} \tilde{c}_t^P$ and:

$$u'\left(\tilde{c}_t^p\right) = \beta E_t \left(1 + \tilde{r}_{t+1}\right) u'\left(\tilde{c}_{t+1}^p\right) \tag{66}$$

where $1 + \tilde{r}_t$ is the period t marginal productivity of capital $1 + \tilde{r}_t = \mu A_t \tilde{K}_t^{\mu-1} L^{1-\mu} + 1 - \lambda$

3.7 Steady state

The next proposition presents steady-state properties to show the conditions under which the conjecture equilibrium structure is valid.

Proposition 7 *1) The steady-state capital stock and aggregate consumption are the same in the market and first-best allocation.*
2) When $\pi^* = 0$, Conjecture 1 is fulfilled, and $c^{PU*} < c^{PA*}$ and $c^{NU*} < c^{NE*}$.

Proof. The proof is simple. First, the Euler equations (48) and (66) in steady state imply $1 + \tilde{r}^* = 1 + \tilde{r}^* = \frac{1}{\beta}$. As a direct consequence, the steady-state level of capital stock is optimal in the market economy, $K^* = \tilde{K}^*$, and aggregate consumption (which can be deduced from the goods market equilibrium) is optimal, as in the simple model.
Second, under Conjecture 62, the two Euler equations (36) and (50) imply:

$$\frac{c^{PU*}}{c^{PA*}} = \left(\left[\frac{1+\pi^*}{\beta} - 1 + T\right]\frac{1}{T}\right)^{-\frac{1}{\sigma}} \text{ and } \frac{c^{NU*}}{c^{NE*}} = \left(\left[\frac{1+\pi^*}{\beta} - \alpha\right]\frac{1}{1-\alpha}\right)^{-\frac{1}{\sigma}}.$$

Plugging these expressions in (63) and (64), one finds that the two conditions (63) and (64) are fulfilled when $1 + \pi^* > \beta$ (i.e. the economy is not at the Friedman rule), what includes the case $\pi = 0$. ■

When $1 + \pi^* > \beta$ then $c^{PU*} < c^{PA*}$ and $c^{NU*} < c^{NE*}$. Indeed, in this case it is costly to self-insure using money because of inflation. Households thus rationally choose to experience a fall in consumption in case of a bad idiosyncratic shock. The central planner choosing the value π^* faces a trade-off between insurance and redistribution, and the optimal value of π^* in the Ramsey problem will depend on ω_p. As in the simple model and as a normalization, I will choose the Pareto weight ω_p such that the Ramsey problem delivers an optimal steady-state net inflation rate equal to 0, when one constrains the model to deliver a realistic amount of lack of insurance (see below).

3.8 Calibration and results

The period is a quarter. Preference parameters are set to standard values. The discount factor is $\beta = 0.99$ and the curvature of the utility function is $\sigma = 2$ (Hall, 2010). The production function is such that the capital share is $\mu = .36$ and the depreciation rate is $\lambda = 0.025$ (Cooley and Hansen, 1989 among others). The discount factor determines the steady-state interest rate $1 + r^* = 1/\beta$, with equation (48). This and the depreciation rate determine the steady-state capital stock and the steady-state wage rate w^* per efficient unit.

Concerning the labor market, a quarterly job-separation rate and job-finding rate is estimated using Shimer (2005) methodology. The quarterly job-separation rate is 5%, such that $\alpha = 0.95$, and the quarterly job-finding rate is 79%, such that $\rho = 0.21$. The replacement rate is calibrated to match the average money holdings of households in the Bottom 50% of the income distribution. It implies a replacement rate of 0.46, which is close to the one used by Shimer (2005). Concerning inequality in income, I take $\kappa = 4.42$ to match the ratio of the income of the Top 50% over that of the Bottom 50% (see the targets below).

Two parameters, α^f and ρ^f, concern the participation structure in financial markets. To my knowledge there is no direct estimation of the participation frequency of households in financial markets. I follow the strategy of Alvarez et al. (2009) which is to calibrate participation frequency to match some monetary moments of the data. First, I set $\rho^f = 0.5$ and $\alpha^f = 0.85$ to match two targets. The first one is the ratio of money over income of the top 50% households. The second one is an average fall in consumption of households transiting from employment to unemployment of 5%. This value is in line with the finding of Gruber (1997) of 7%. As a consequence, the model generates a realistic amount of uninsurable risk for money holders, which is key for welfare analysis. This calibration strategy implies that 77% of the P-households participate in financial markets each period, and the probability not to participate next period, when participating, is 15%.

The process for technology is set to standard values. The persistence of technology shock is set to $\rho^a = 0.95$ and the standard deviation is $\sigma_a = 1\%$ (Cooley and Hansen, 1989). The last parameter to be determined is the Pareto weights ω_p. I choose ω_p such that the optimal inflation in the steady-state Ramsey problem is 0. This is a normalization, as we set parameter values such that the model delivers a realistic amount of lack of self-insurance (measured by the fall in consumption when experiencing a fall in income). Solving the model numerically, one finds $\omega_p = 3.33$. The next table presents the parameter values.

Population (%) and Pareto weight				Preferences and technology					
	Ω^N	Ω^P	ω_p	β	σ	μ	λ	ρ^a	σ_a
	50	50	3.33	.99	2	.36	.025	.95	.01
Income structure				Uninsurable risk					
κ	δ^N/w			α	ρ	α^f	ρ^f		
4.42	0.46			.95	.21	.85	.5		

Table 2: Parameter values. See text for description.

As a summary, the next table presents the outcome for the four targets, which are used to calibrate the four parameters κ, ρ^f, α^f and δ^N.

	Data	Model
Annual income T50/B50	5.3	5.3
Money/Annual Income T50	12.7	12.7
Money/Annual Income B50	9.5	9.5
Fall in consump. for unempl.	7%	5%

Table 3: Calibration targets.

The model outcome for all relevant variables is provided in the Online Appendix OA5. The model matches well (by construction) the money distribution.

Model resolution. The gain of the assumptions made above is that the model is easy to solve numerically. In particular, to derive the optimal Ramsey policy, I consider a second-order approximation of the welfare objective and a first-order approximation of the constraints. Standard linear-quadratic methods allow deriving the optimal monetary policy of the central planner. It is then possible to compute the steady-state inflation rate generated by the solution of the Ramsey problem, as a function of the Pareto weight. I iterate over the Pareto weight until the steady-state optimal inflation rate is 0.

3.8.1 The effect of optimal monetary policy

To understand the trade-offs faced by monetary policy in this environment, the optimal monetary policy after a positive technology shock is now derived. Three economies are compared. The first one is the Ramsey allocation, where the planner solves the Ramsey problem (60). The second one is the first-best allocation, where the central planner is unconstrained and can implement the first best allocation. The third one is an inactive-policy allocation where monetary policy is inactive, i.e. the nominal money stock is constant. In this last economy, we impose that m_t^{CB} is 0. As the

Pareto weights have been chosen such that the optimal steady-state inflation rate solving the Ramsey problem is 0, the steady-state inflation is the same in the Ramsey and the inactive-policy allocations. As a consequence, the gain of an active monetary policy is only the result of its ability to affect the business cycle and is not the outcome of a reduction in steady-state distortions.

We plot in Figure 5 the main variables in the three economies after the same technology shock.

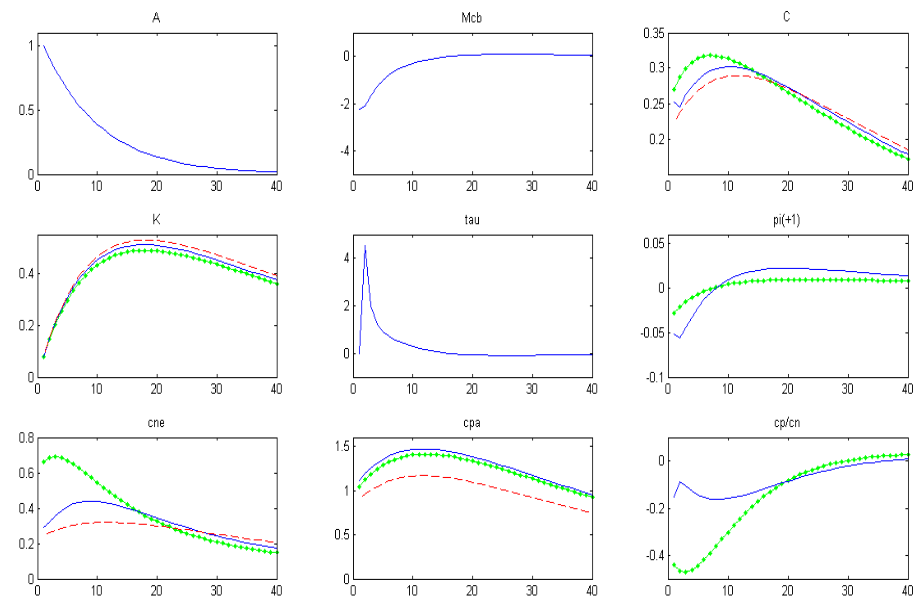

Fig 1: Outcomes of the three economies after the same technology shock; in percentage deviation from steady state for technology (A), aggregate consumption (C), capital (K), consumption levels (cne and cpa) and consumption inequality (cp/cn); in percent for money creation (Mcb), taxe (tau).
The green dotted line is the market economy with inactive monetary policy ($m^{CB} = 0$). The blue solid line is the market economy with a constrained efficient monetary policy. The red dotted line is the first-best allocation.

The first panel, at the top left, presents the technology shock A_t, as percentage deviation from the steady-state value. The second panel presents optimal money created m_t^{CB} when the central planner solves the Ramsey program. Optimal monetary policy is countercyclical. It reduces the money in circulation after a positive technology shock. Although agggregate consumption is increasing in the three economies (as represented in the third panel of the first line), optimal monetary policy reduces the increase in consumption compared to the inactive-policy allocation. The first-best increase in consumption is even smaller. Optimal monetary policy generates an additional increase in the capital stock (compared to the inactive-policy allocation), although less than its level in the first-best allocation. Optimal monetary policy increases the capital stock by 5% at peak. It thus generates "forced saving". Panel 5 are taxes and Panel 6 is the expected inflation rate.

Comparing Ramsey and inactive-policy allocations, one can see that the contractionary monetary policy is a transfer from $N-$households to $P-$households. In the Ramsey allocation, the consumption of employed $N-$households (who are the non-participaitng money holders) is c^{NE} and is represented in the Panel 7. This consumption is lower than the one in the inactive-policy allocation. In addition, the consumption of P-households who are participating in financial markets (c^{PA} represented in Panel 8) is higher in the Ramsey allocation than in the inactive-policy allocation. These P-households save more after the technology shock in the Ramsey allocation than in the inactive-policy allocation, as can be seen from the path of the aggregate shock.

Optimal monetary policy thus increases the capital stock after such a shock, but undesirable redistributive effects limit the ability to restore the first best capital dynamics. To see this, Panel 9 plots consumption inequalities as the ratio of total consumption of participating households over total consumption of non-participating households. The average consumption of participating households is higher than that of non-participating households (who don't hold the capital stock). As a consequence, the graph shows that consumption inequality decreases less in the Ramsey allocation than in the inactive-policy allocation. In other words, the contractionary monetary policy contributes to an increase in inequality, which is consistent with recent empirical findings (Coibion et al. 2012).

Before presenting the welfare implications, the next Table reports the second-order moments of aggregate variables for the three allocations.

Economies	Variables		
	K (%)	C^{tot}(%)	$corr(C^{tot}_{t-1}, C^{tot}_{t})$
Inactive-policy	3.03	13.1	0.988
Ramsey	3.15	13.0	0.990
First-best	3.26	13.0	0.992

Table 4: Second-order moments of key variables.

The volatility of the capital stock is higher in the Ramsey allocation compared to the inactive-policy allocation, but it remains lower than in the first best allocation. This implies that the volatility of aggregate consumption falls in the Ramsey allocation compared to the inactive-policy allocation. In other words, the capital stock does not react enough (and consumption reacts too much) to the technology shock when monetary policy is inactive. In addition, the autocorrelation of aggregate consumption increases in the Ramsey allocation, compared to the inactive-policy allocation: More volatile capital stock translates into smoother aggregate consumption, because it is a way to save more in good times (and less in bad times), as can be seen from Figure 1.

3.8.2 Welfare gains of an active monetary policy

One can compute the welfare gains of an active monetary policy (i.e. solving the Ramsey program) compared to an inactive monetary policy, for both P and N households. I follow the standard measure of consumption equivalent in the heterogeneous-agents literature. I first compute the average welfare for the inactive allocation of $N-$households by simulating the economy with inactive policy for 10,000 periods. I then compute the ex-ante welfare as $W_{IN}^{N}=\sum_{t=0}^{\infty}\beta^{t}\left(n^{NE}u\left(c_{IN,t}^{NE}\right)+n^{NU}u\left(c_{IN,t}^{NU}\right)\right)$ (where IN stands for the inactive-policy allocation). Similarly, I compute the ex-ante welfare of P households in the inactive allocation $W_{IN}^{P}=\sum_{t=0}^{\infty}\beta^{t}\left(n^{PA}u\left(c_{IN,t}^{PA}\right)+n^{PU}u\left(c_{IN,t}^{PU}\right)\right)$.

I then compute the ex-ante welfare for the Ramsey allocation by simulating the economy where the central planner solves the Ramsey problem for 10,000 periods. I can similarly compute the ex-ante welfare of the P and N households in the Ramsey allocation. This gives W_{Ramsey}^{N} and W_{Ramsey}^{P}. The consumption equivalent is the average increase in consumption that N and P households would need to enjoy in the inactive-policy allocation to have the same ex-ante welfare as in the Ramsey allocation. Mathematically, one computes Δ^{N} and Δ^{P} such that:

$$\sum_{t=0}^{\infty}\beta^{t}\left(n^{NE}u\left(c_{IN,t}^{NE}\left(1+\Delta^{N}\right)\right)+n^{NU}u\left(c_{IN,t}^{NU}\left(1+\Delta^{N}\right)\right)\right)=W_{Ramsey}^{N}$$

$$\sum_{t=0}^{\infty}\beta^{t}\left(n^{PA}u\left(c_{R,t}^{PA}\left(1+\Delta^{P}\right)\right)+n^{PU}u\left(c_{R,t}^{PU}\left(1+\Delta^{P}\right)\right)\right)=W_{Ramsey}^{P}$$

One finds that $\Delta^{N}=0.39\%$ and $\Delta^{P}=0.19\%$. Optimal monetary policy increases the welfare of both types of agents, but it increases the welfare of $N-$households more than the welfare of $P-$households. Monetary policy increases consumption smoothing (which is not optimal because some households do not participate in financial markets). This benefits relatively more $N-$households, who do not have access to financial markets. Note that the welfare gains are much higher than the gains from eliminating business cycles in representative agent economies.

3.8.3 Financial returns and the return on money

Finally, the key mechanism in the paper is that, for many households, the opportunity cost of consumption is the return on money and not the return on financial assets. One can now directly compute the empirical correlation between these two returns and compare it to the ones obtained in the model. To compute the return on financial assets, I consider the quarterly returns on the Standard and Poor's Composite Stock Index, which is the standard index to compute stock market returns[17], from 1960 to 2014.The returns are computed as $R^s_{t+1} = (P_{t+1} - P_t + D_{t+1})/P_t$, where P_t is the real stock price index in period t, and D_t is the sum of the real dividends paid the 3 months before quarter t[18]. The net return on money is simply $(1/\Pi_{t+1} - 1)$, where Π_{t+1} is the gross inflation rate measured with the CPI index. I find an empirical correlation between the return on money and the return on financial assets equal to 0.14.

For the model, I compute the correlation between the return on claims on the financial stock and the return on money. I compute the correlation in the case of the Ramsey allocation, thus assuming that monetary policy was optimally set during this period (to obtain meaningful numbers). The model generates a correlation equal to 0.45 (instead of 0.14 in the data). As there is only one shock in the model, one cannot expect this simple model to match the empirical correlation quantitatively. Nevertheless, the model delivers a positive and low correlation as a general equilibrium outcome, as in the data. In the model, a positive TFP shock increases financial returns and the real wage. As a consequence, it increases money demand, which decreases the price level. It thus increases the expected return on money, generating a positive correlation. Interestingly, a positive (and possibly high) correlation between the return on money and on financial assets is not evidence of the absence of frictions in financial markets. Instead, the empirical distribution of money may be a better indication of such frictions.

4 Concluding remarks

This paper derives some macroeconomic implications of new developments concerning money demand. These developments conclude that financial frictions such as limited participation in financial markets are key to understanding money holdings. A direct implication is that the marginal return on savings is the return on money for many households, and not the marginal product of capital. The distortions generated by this simple friction are surprisingly complex. Investment can be either too high or too low in the business cycle compared to the first best allocation. It has been shown in the theoretical model that the market economy behaves as if the discount factor is too high compared to the one used by the central planner, because the social return of capital accumulation is not correctly perceived by the subgroup of market participants. In the more general model, it has been shown that the market economy underinvests after a typical technology shock, when monetary policy is inactive. In this setup, monetary policy has to balance a positive effect on capital accumulation and

[17] I use the stock market return to be consistent with the model. Note that bond holding is even more concentrated than stock holding, as only 20% of households hold bonds either directly or indirectly (Bricker et al., 2014).

[18] I generated the same return using earnings instead of dividends. The results are quantitatively similar.

undesirable redistributive effects.

The additional interest of the general model is to present a tractable incomplete insurance market model with limited participation, which generates a simple but realistic distribution of money. The model is simple enough to perform a welfare analysis with aggregate shocks. Admittedly, this model is not a full-fledged quantitative analysis of the business cycle, as many other relevant ingredients for business cycle analysis are missing. An obvious path for future work is to introduce other frictions in this model, such as nominal frictions or a search-and-matching model of the labor market, to study their interaction with limited participation. These interactions may help to think about important trade-offs for monetary policy. For instance, introducing nominal frictions, monetary policy will generate an additional aggregate-demand effect due to a redistribution effect across heterogeneous households. This analysis is left for future work.

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A Proof of Proposition 1

Using the equations for the optimal program (Equations 17 and 15), one finds $1+\tilde{r}^* = \frac{1}{\beta}$ and $\tilde{K}^* = L(\Omega)(\beta\mu)^{1/(1-\mu)}$, the same value as in the market economy. As a consequence, $K^* = \tilde{K}^*$ and $Y = \tilde{Y}^*$, and total consumption is the same in both economy $C^{tot*} = \tilde{C}^{tot*}$. The central planner allocation implies $\tilde{c}^{n*}/\tilde{c}^{p*} = \omega_p^{-\frac{1}{\sigma}}$. The market allocation is, when $\pi = 0$:

$$c^{p*} = \left(\mu(1-\beta)\frac{1-\Omega/2}{1-\Omega} + 1 - \mu\right)(\beta\mu)^{\frac{\mu}{1-\mu}} \text{ and } c^{n*} = w^* = (1-\mu)(\mu\beta)^{\frac{\mu}{1-\mu}}$$

As total consumption is the same in the market economy and for the optimal allocation, a necessary and sufficient condition to have $c^{n*} = \tilde{c}^{n*}$ and $c^{p*} = \tilde{c}^{p*}$ is $\tilde{c}^{p*}/\tilde{c}^{n*} = c^{p*}/c^{n*}$. Using the three previous equations, this condition can be written as the one given by equality (18).

B Proof of Propositions 2

Consider the dynamic system (19)-(20). Using (20), one can substitute $\hat{c}_t^p$ in (19), to obtain a single equation in $\hat{K}_t, \hat{K}_{t+1}$ and $\hat{K}_{t+2}$. Using the method of unknown coefficients, one finds that the capital stock has the form:

$$\hat{K}_{t+1} = B(\sigma, \theta(\Omega))\hat{K}_t + D^a(\sigma, \theta(\Omega), \rho^a)a_t \quad (67)$$

where ρ^a is the persistence of the technology shock, and where :

$$\begin{aligned} B(\sigma,\theta) &= \frac{1}{2\sigma}\left((1-\mu(1-\sigma))\theta+\mu+\sigma-1\right) \\ &\quad -\frac{1}{2}\sqrt{\frac{1}{\sigma^2}\left((1-\mu(1-\sigma))\theta+\mu+\sigma-1\right)^2-4\theta\mu} \\ D^a(\sigma,\theta,\rho) &= \frac{\theta+\frac{\rho}{\sigma}\left(\theta(1-\sigma)-1\right)}{\frac{1}{\sigma}\left((1-\mu(1-\sigma))\theta+\mu+\sigma-1\right)-B(\sigma,\theta)-\rho} \end{aligned} \tag{68}$$

Comparing (19)-(20) and (22)-(23), on can observe that the optimal and market allocations are the same when $\Omega=0$ because $\theta(0)=1/(\mu\beta)$. One thus directly find the optimal low of motion of the capital stock:

$$\widehat{\tilde{K}}_{t+1}=\tilde{B}\widehat{\tilde{K}}_t+\tilde{D}^a a_t \text{ with } \tilde{B},\tilde{D}^a>0 \tag{69}$$

with $\tilde{B}=B(\sigma,\theta(0))$ and $\tilde{D}^a=D^a(\sigma,\theta(0),\rho)$.

Moreover, when $\sigma=1$, whatever the value of θ (and thus of Ω), one finds $B(1,\theta)=\mu$ and $D(1,\theta,\rho)=1$. As the consequence, the dynamics of the capital stock is the same in both economies. It is then easy to show that the consumption of both P and $N-$households is the same in both economies (using the good-market equilibrium), what concludes the proof.

C Proof of the Proposition 3

Assume that $\sigma=1+\varepsilon$ with ε small such that a first order expansion in ε relevant. One finds

$$\begin{aligned} B(\varepsilon,\theta) &= \mu+(\mu-1)(\theta-1)\frac{1}{2}\left(1-\frac{1}{(\theta+\mu)(\theta-\mu)}\right)\varepsilon \\ D^a(\varepsilon,\theta,\rho) &= 1+\frac{\theta-1}{\theta-\rho}\left((1-\mu)\frac{\theta}{\theta-\mu}-\rho\right)\varepsilon \end{aligned}$$

Using (21), the dynamic of consumption can be written as $\frac{\partial\widehat{c}^n}{\partial\varepsilon^a}=1$ and plugging the expression of $\hat{K}_{t+1}$ given by (67), in the budget constraint of $P-$households (20), one finds $\hat{c}^p_t=\frac{\theta-D^a}{\theta-1}a_t+\frac{\theta\mu-B}{\theta-1}\hat{K}_t$. Hence

$$\frac{\partial\widehat{c}^p}{\partial\varepsilon^a}=\frac{\theta-D^a(\varepsilon,\theta,\rho)}{\theta-1} \tag{70}$$

Define $\overline{\rho}^1$ as $\overline{\rho}^1=\theta\frac{1-\mu}{\theta-\mu}$. As $\theta>1$, we have $0<\overline{\rho}^1<1$.

When $\rho>\overline{\rho}^1$, one can easily check that then $D^a(\varepsilon,\theta,\rho)<1$, and one finds that when $\rho<\overline{\rho}^1$, we have $D^a(\varepsilon,\theta,\rho)>1$. What concludes the proof.

D Proof of the Proposition 4

As the solution of the optimal program for the $P-$households is the same as the one in the market economy for $\theta(0)$, one has from (70):

$$\frac{\partial \widehat{c}^p}{\partial \varepsilon^a} = \frac{\theta(\Omega) - D^a(\varepsilon, \theta(\Omega), \rho)}{\theta(\Omega) - 1} \text{ and } \frac{\partial \widehat{\tilde{c}}^p}{\partial \varepsilon^a} = \frac{\theta(0) - D^a(\varepsilon, \theta(0), \rho)}{\theta(0) - 1}$$

Recall that $\theta(\Omega)$ is decreasing in Ω, hence $\theta(\Omega) < \theta(0)$ for $\Omega > 0$. Moreover, from (67) and (69), one has:

$$\frac{\partial \widehat{\hat{K}}}{\partial \varepsilon^a} = D^a(\varepsilon, \theta(0), \rho) \text{ and } \frac{\partial \hat{K}}{\partial \varepsilon^a} = D^a(\varepsilon, \theta(\Omega), \rho)$$

The proof relies on the following Lemma:

Lemma 8 *There is a $\bar{\rho}^2, 0 < \bar{\rho}^2 < \bar{\rho}^1$, such that*

$$\text{If } \rho < \bar{\rho}^2,\ D^a(\sigma, \theta(\Omega), \rho) < \tilde{D}^a(\sigma, \theta(0), \rho)$$
$$\text{If } \rho > \bar{\rho}^2,\ D^a(\sigma, \theta, \rho) > \tilde{D}^a(\sigma, \theta(0), \rho)$$

Proof of the Lemma. For small ε :

$$D^a(\sigma, \theta, \rho) = 1 + \frac{\theta - 1}{\theta - \rho}\left((1-\mu)\frac{\theta}{\theta - \mu} - \rho\right)\varepsilon$$

Using the previous expression, one finds:

$$\frac{1}{\varepsilon}\frac{\partial}{\partial \theta} D^a(\sigma, \theta, \rho) = \left(\frac{1-\rho}{\theta - \rho}\left((1-\mu)\frac{\theta}{\theta-\mu} - \rho\right) - (\theta - 1)(1-\mu)\frac{\mu}{(\theta-\mu)^2}\right)\frac{1}{\theta - \rho}$$

Define $F(\rho) \equiv \frac{1-\rho}{\theta-\rho}\left((1-\mu)\frac{\theta}{\theta-\mu} - \rho\right) - (\theta-1)(1-\mu)\frac{\mu}{(\theta-\mu)^2}$. From the previous expression, the sign of $\frac{\partial}{\partial\theta} D^a(\sigma, \theta, \rho)$ is the sign of $F(\rho)$.

If $\rho > \theta\frac{1-\mu}{\theta-\mu}$ then $F(\rho) < 0$.

If $\rho < (1-\mu)\frac{\theta}{\theta-\mu}$ then $F(\rho)$ is decreasing and continuous in ρ., $F(0) > 0$ and $F\left((1-\mu)\frac{\theta}{\theta-\mu}\right) < 0$. As a consequence, there is a $\bar{\rho}^2, 0 < \bar{\rho}^2 < \rho^1 = (1-\mu)\frac{\theta}{\theta-\mu} < 1$, such that $F(\rho) > 0$ if $\rho < \bar{\rho}^2$ and $F(\rho) < 0$ if $\rho > \bar{\rho}$. As a consequence

$$\frac{\partial}{\partial\theta} D^a(\sigma, \theta, \rho) > 0 \text{ if } \rho < \bar{\rho}^2$$
$$\frac{\partial}{\partial\theta} D^a(\sigma, \theta, \rho) < 0 \text{ if } \rho > \bar{\rho}^2$$

Hence, if$\rho < \bar{\rho}^2$, then $D^a(\sigma, \theta(\Omega), \rho) < D^a(\sigma, \theta(0), \rho)$, and the reverse when $\rho > \bar{\rho}^2$, what concludes the proof of the Lemma.

To conclude the proof of the proposition, one can use the lemma to rank the impact response for $\widehat{c}^p$, $\widehat{\tilde{c}}^p$, $\widehat{\hat{K}}$ and $\hat{K}$.

E Proof of Proposition 5

Define as $\tilde{g}\left(A_t, \tilde{K}_t\right)$ the optimal decision rule of the central planner : $\tilde{K}_{t+1} = \tilde{g}\left(A_t, \tilde{K}_t\right)$, solving the program (15) and (17) and which is uniquely defined by standard dynamic programming argument. Assume that in the market economy the money supply follows the rule $m_t^{CB} = H\left(\Omega, A_t, K_t\right)$ where,

$$H\left(\Omega, A_t, K_t\right) \equiv \left(\left((1-\mu)\frac{1-\Omega}{1-\frac{\Omega}{2}} + \mu\right)\left(K^*\right)^{\mu} L^{1-\mu} - K^*\right)\left(\frac{A_t\tilde{K}_t^{\mu}L^{1-\mu} - \tilde{g}\left(A_t, \tilde{K}_t\right)}{\left(K^*\right)^{\mu}L^{1-\mu} - K^*}\right)$$
$$- \left((1-\mu)\frac{1-\Omega}{1-\frac{\Omega}{2}} + \mu\right) A_t K_t^{\mu} L^{1-\mu} + \tilde{g}\left(A_t, K_t\right)$$

Although this expression is complex, it is only a function of the past state variables K_t,and on the current technology shock A_t. I now shown that the first best allocation is a solution of the program of all agents in the market economy, when monetary policy follows the previous rule. As a consequence, optimal monetary policy can implement the first best[19]. The proof is done in two steps.

First, using the budget constraint of participating households (3) and (9), one finds that the budget constraint of participating households can be written as a simple system in c_t^p and K_t (plugging the expression of H and using $\tilde{c}^{p*} = c^{p*}$)

$$\left(\frac{\Omega}{2}\left(\omega_p\right)^{-\frac{1}{\sigma}} + (1-\Omega)\right) c_t^p = A_t K_t^{\mu} L^{1-\mu} - \tilde{g}\left(A_t, K_t\right) + \frac{\frac{\Omega}{2}\left(\omega_p\right)^{-\frac{1}{\sigma}} + (1-\Omega)}{1-\Omega}\left(\tilde{g}\left(A_t, K_t\right) - K_{t+1}\right)$$

Second, we can show that the optimal decision rule $K_{t+1} = \tilde{g}\left(A_t, K_t\right)$ (the one derived from the solution of the program of the central planner) is a solution to the problem of participating households. Indeed, the program of these households can be written as

$$u'\left(c_t^p\right) = \beta E_t\left(\mu A_{t+1} K_{t+1}^{\mu-1} L^{1-\mu}\right) u'\left(c_{t+1}^p\right)$$
$$K_{t+1} + \left(\frac{\Omega}{2}\left(\omega_p\right)^{-\frac{1}{\sigma}} + (1-\Omega)\right) c_t^p = A_t K_t^{\mu} L^{1-\mu} + \frac{1}{2}\frac{\Omega}{1-\Omega}\left(\omega_p\right)^{-\frac{1}{\sigma}}\left(\tilde{g}\left(A_t, K_t\right) - K_{t+1}\right)$$

One recognizes the program of the central planer (17) and the budget constraint given in Section 2.3, with an extra term at the right hand side $\frac{1}{2}\frac{\Omega}{1-\Omega}\left(\omega_p\right)^{-\frac{1}{\sigma}}\left(\tilde{g}\left(A_t, K_t\right) - K_{t+1}\right)$, which is nul when $K_{t+1} = \tilde{g}\left(A_t, K_t\right)$. As a consequence, if $K_{t+1} = \tilde{g}\left(A_t, K_t\right)$ is a solution of the central planner program, it is also a solution of the program of $P-$households in the limited-participation economy. Hence, $c_t^p = \tilde{c}_t^p$ and $K_t = \tilde{K}_t$ and $c_t^n = \tilde{c}_t^n$ by the goods market equilibrium.

[19] It has been check that the first best allocation is the only possible equilibrium in a first order approximation of the dynamics. In other words, the equilibrium is locally unique.

Online Appendix (OA)

This Online Appendix presents the following additional results

1. Summary of the simple model
2. An extension to inside money
3. Summary of the general model
4. Steady-state existence condition
5. General model outcome

OA1 - Summary of the simple model

$$m_t^n = w_t + \tau_t$$
$$c_t^n = \frac{m_{t-1}^n}{1+\pi_t} + \tau_t$$

$$u'\left(c^p\right) = \beta E_t \left(1+r_{t+1}\right) u'\left(c_{t+1}^p\right)$$
$$k_{t+1}^p + c_t^p = w_t + \tau_t + \left(1+r_t\right) k_t^p$$

$$w_t = \left(1-\mu\right) A_t K_{t-1}^{\mu} L^{-\mu}$$
$$1 + r_t = \mu A_t K_t^{\mu-1} L^{1-\mu}$$
$$\tau_t = m_t^{CB}$$

$$m_t^n = \frac{m_{t-1}^n}{1+\pi_t} + \frac{2}{\Omega} m_t^{CB}$$
$$\left(1-\Omega\right) k_t^p = K_t$$
$$\left(1-\Omega\right) c_t^p + \frac{\Omega c_t^n}{2} + K_{t+1} = A_t K_{t-1}^{\mu} L^{1-\mu}$$
$$L = 1 - \Omega/2$$

$$A_t = e^{a_t}, \text{where } a_t^a = \rho^a a_{t-1} + \varepsilon_t^a$$

The equations characterizing the optimal allocation can be derived taking $\Omega = 0$.

OA2 - Inside money

To prove that the distortions identified in the benchmark economy don't rely on imposing outside money in the economy and that they exist with inside money, it is now assumed that $N-$households have access to financial markets, but with a proportional cost which can be arbitrarily low. it is shown that the allocation is the same as in the benchmark equilibrium, and that it exhibits the same distortions, although $N-$households hold assets, which can be called inside liquidity.

More specifically, it is now assumed that $N-$households save in a representative fund, which can invest an amount $\frac{\Omega}{2}\check{k}^n_{t+1}$ in the capital stock (the term $\Omega/2$ is a normalization to simplify the algebra), but which faces transaction cost. I now use tilda to denote the value of variables with inside money.

For each unit invested in the fund, the fund yield a return $1+\lambda r_t$ with $\lambda \leq 1$, instead of $1+r_t$. The transaction cost λ covers the information-processing cost of the fund and it can be arbitrarily close but less than 1. In each period, the number $\Omega/2$ of N-households invest their after-tax income $\check{w}_t - \check{\tau}_t$, and $\Omega/2$ $N-$households receive some income $\check{a}^n_t$ from the fund. The program of the fund is

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \frac{\Omega}{2} u\left(\check{c}^n_t\right)$$

The budget constraint is now

$$\begin{aligned} \check{k}^n_{t+1} + \check{a}^n_t &= \left(1+\lambda \check{r}_t\right)\check{k}^n_t + \left(\check{w}_t - \check{\tau}_t\right) \\ \check{c}^n_t &= \check{a}^n_t - \tau_t \\ \check{k}^n_{t+1} &\geq 0 \end{aligned}$$

where $\check{a}^n_t$ is now the before-tax income of $N-$households. As money is a dominated asset when $\pi = 0$, the fund does not hold money. The next Proposition shows that the allocation in this economy is the same as the one under limited participation.

Proposition 9 *If $\lambda < 1$,the allocation in the inside-money economy is the same one as in the limited-participation economy with outside money when $m^{CB}_t = 0$.*

Proof : We show that the steady-state allocation is the same in the inside-money economy and in the limited-participation economy. As public spending is normalized to 0, the level of taxes is $\tau_t = 0$. The first-order condition for the program of the fund is:

$$u'\left(\check{c}^n_t\right) = \beta\left(1+\lambda r\right) u'\left(\check{c}^n_{t+1}\right)$$

In steady state, one can check that $1+\lambda r = 1-\lambda+\lambda/\beta < 1/\beta$. As a consequence, the assets held by the funds decrease until $\check{k}^{n*} = 0$ (the argument is the same as in Kiyotaki and Wright, 1998) The steady state in the inside money equilibrium is:

$$\check{c}^{n*} = w^*$$

As taxes are 0, one easily finds that all steady-state values are the same in both economies. Finally, if the support of technology shock is small enough, one finds that the credit constraint is binding in the dynamic economy. It implies that the budget constraint is now:

$$\check{c}_t^n = \check{w}_t$$

In the limited-participation economy, we have the same equation. Indeed:

$$c_t^n = \frac{m_{t-1}^n}{1+\pi_t} = m_t^n = w_t$$

As all the other equations are the same, it is direct to show that the dynamic of two economies is represented by the same equations.

OA3 - Summary of the General Model

To ease the understanding of the general model, I now provide the whole set of non-linear equations for the equilibrium under consideration. Existence conditions are given in the next Section. The equations concerning $N-$households are:

$$\begin{aligned} u'\left(c_t^{NE}\right) &= \beta E_t \left[\alpha u'\left(c_{t+1}^{NE}\right) + (1-\alpha)\, u'\left(c_{t+1}^{NU}\right)\right] \frac{1}{1+\pi_{t+1}} \\ c_t^{NU} &= \frac{1}{1+\pi_t}\frac{1-\rho}{\alpha} m_t^{NE} + \delta^N - \tau_t \\ c_t^{NE} + \frac{1}{\alpha} m_{t+1}^{NE} &= \frac{m_t^{NE}}{1+\pi_t} + w_t - \tau_t \end{aligned}$$

The equations concerning $P-$households are:

$$\begin{aligned} u'\left(c_t^{PA}\right) &= \beta E_t \left(1+r_{t+1}\right) u'\left(c_{t+1}^{PA}\right) \\ u'\left(c_t^{PA}\right) &= \beta E_t \frac{1+i_t}{1+\pi_{t+1}} u'\left(c_{t+1}^{PA}\right) \\ u'\left(c_t^{PA}\right) &= \beta E_t \left[(1-T)\, u'\left(c_{t+1}^{PA}\right) + T u'\left(c_{t+1}^{PU}\right)\right] \frac{1}{1+\pi_{t+1}} \\ c_t^{PA} + k_{t+1}^{PA} + b_{t+1}^{PA} + \frac{1}{1-T} m_{t+1}^{PA} &= \kappa \frac{n w_t + n^{PA}(1-n)\,\delta^N}{n + n^{PA}(1-n)} - \tau_t + \frac{m_t^{PA}}{1+\pi_t} + (1+r_t)\, k_t^{PA} + \frac{1+i_{t-1}}{1+\pi_t} b_t^{PA} \\ c_t^{PU} &= \kappa\delta^N - \tau_t + \frac{1}{1+\pi_t}\frac{1-\rho\rho^f}{1-T} m_t^{PA} \end{aligned}$$

First order condition of the firm:

$$w_t = (1-\mu) A_t K_t^{\mu} L^{-\mu}$$
$$r_t + \lambda = \mu A_t K_t^{\mu-1} L^{1-\mu}$$

Budget of the State and fiscal rule:

$$\tau_t = \frac{1+i_{t-1}}{1+\pi_t} m_{t-1}^{CB}$$

Law of motion of money and market equilibria:

$$m_{t+1}^{tot} = \frac{m_t^{tot}}{1+\pi_t} + m_t^{CB}$$
$$m_t^{tot} = \frac{\Omega^N}{\alpha} n m_t^{NE} + \frac{\Omega^P}{1-T} n^{PA} m_t^{PA}$$
$$\Omega^P n^{PA} b_{t+1}^{PA} + m_t^{CB} = 0$$
$$L = n\left(\Omega^N + \Omega^P \kappa\right)$$

Technology shock:

$$A_t = e^{a_t}, \text{where } a_t^a = \rho^a a_{t-1} + \varepsilon_t^a$$

The evolution of m_t^{CB} is the missing equation. It is given either by the solution of the Ramsey problem or by the assumption $m_t^{CB} = 0$.

Existence conditions are

$$u'\left(c_t^{NU}\right) > \beta E_t\left[(1-\rho) u'\left(c_{t+1}^{NE}\right) + \rho u'\left(c_{t+1}^{NU}\right)\right] \frac{1}{1+\pi_{t+1}}$$
$$u'\left(c_t^{PU}\right) > \beta E_t\left[\left(1-\rho\rho^f\right) u'\left(c_{t+1}^{PA}\right) + \rho\rho^f u'\left(c_{t+1}^{PU}\right)\right] \frac{1}{1+\pi_{t+1}}$$

OA4 - Steady-state Existence condition for a given π

The two previous Euler equations give

$$\frac{c^{PU}}{c^{PA}} = \left(\left[\frac{1+\pi}{\beta} - 1 + T\right]\frac{1}{T}\right)^{-\frac{1}{\sigma}} \text{ and } \frac{c^{NU}}{c^{NE}} = \left(\left[\frac{1+\pi}{\beta} - \alpha\right]\frac{1}{1-\alpha}\right)^{-\frac{1}{\sigma}}.$$

$$\left(\frac{c^{PU}}{c^{PA}}\right)^{-\sigma} = \left[\frac{1+\pi}{\beta} - 1 + T\right]\frac{1}{T} \text{ and } \left(\frac{c^{NU}}{c^{NE}}\right)^{-\sigma} = \left[\frac{1+\pi}{\beta} - \alpha\right]\frac{1}{1-\alpha}$$

Steady-state existence conditions can be written as

$$1 > \beta\left[(1-\rho)\left(\frac{c^{NE}}{c^{NU}}\right)^{-\sigma} + \rho\right]\frac{1}{1+\pi}$$
$$1 > \beta\left[\left(1-\rho\rho^f\right)\left(\frac{c^{PA}}{c^{PU}}\right)^{-\sigma} + \rho\rho^f\right]\frac{1}{1+\pi}$$

or

$$\frac{1+\pi}{\beta} > \frac{(1-\rho)(1-\alpha)}{\frac{1+\pi}{\beta}-\alpha} + \rho$$
$$\frac{1+\pi}{\beta} > \frac{T\left(1-\rho\rho^f\right)}{\frac{1+\pi}{\beta}-1+T} + \rho\rho^f$$

Proposition 10 *The two conditions are fulfiled as soon as* $1+\pi > \beta$*.*

Proof. Define

$$f_\pi(x) = x - \frac{(1-\rho)(1-\alpha)}{x-\alpha} - \rho$$
$$g_\pi(x) = x - \frac{T\left(1-\rho\rho^f\right)}{x-1+T} - \rho\rho^f$$

The two functions are C^1 when $x > 1$. The two conditions can be written as :

$$f_\pi\left(\frac{1+\pi}{\beta}\right) > 0 \text{ and } g_\pi\left(\frac{1+\pi}{\beta}\right) > 0$$

Note that $f_\pi(1) = g_\pi(1) = 0$, and $f'_\pi, g'_\pi > 0$. As a consequence, $f_\pi(x) > 0$ and $g_\pi(x) > 0$ when $x > 1$. Thus $f_\pi\left(\frac{1+\pi}{\beta}\right) > 0$ and $g_\pi\left(\frac{1+\pi}{\beta}\right) > 0$ when $\frac{1+\pi}{\beta} > 1$. ■

OA5 - General Model Outcome

The next Table presents the model outcome, at quarterly frequency, for the parameter values given in the text.

Aggregate quantities, prices and inequalities							
K	n	L	r	w			
96.7	.94	2.55	1%	2.37			
Consumption and money levels							
	c^{NE}	c^{NU}	c^{A}	c^{PE}	c^{PU}	m^{NE}	m^{P}
	2.31	2.10	12.13	12.1	9.01	1.22	4.63

Table : Model outcome