Technical Appendix for "Managing Inequality over the Business Cycles: Optimal Policies with Heterogeneous Agents and Aggregate Shocks"

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In this Technical Appendix, we present additional results and detailed proofs. More precisely:

1. In Section 1, we develop a simple example to illustrate the role of the sign of Lagrange multiplier on Euler equation in the Ramsey program.
2. In Section 2, we derive the first-order conditions related to the Ramsey program.
3. In Section 4, we present a decentralization mechanism in the case of explicit partitions.
4. In Section 3, we report additional moments in level.

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1 Understanding Lagrange multipliers on Euler equations

The analysis of the main text uses Lagrange multipliers on Euler equations and claims that these multipliers can be either positive of negative and that their sign is related to the distortions on the saving incentives (from the planner's point of view). This section provides a very simple example to illustrate this statement.

Consider a two-period endowment economy, where a representative agent has a log utility and a discount factor equal to 1. The agent is initially endowed with one unit of goods but does not receive any income – besides saving payoffs – in the second period. The agent's program can be expressed as:

$$\max_{c_0,c_1,s} \log(c_0) + \log(c_1),$$
$$c_0 = 1 - s,$$
$$c_1 = Rs,$$

where c_t $(t = 0, 1)$ is the consumption at date t and s the saving level. The agent's Euler equation can simply be written as:

$$c_1 = Rc_0. \tag{1}$$

Consider now the planner's program. The only twist is that the planner internalizes the saving externality through a penalty proportional to the saving level and equal to ϕs. The quantity ϕ can be either positive or negative, corresponding respectively to a positive or a negative externality. The Ramsey program of the planner includes the Euler equation (1) and can be written as:

$$\max_{c_0,c_1,s} \log(c_0) + \phi s + \log(c_1)$$
$$c_0 = 1 - s$$
$$c_1 = Rs$$
$$c_1 = Rc_0$$

Introducing the Lagrange multiplier λ on the Euler equation and substituting consumption expressions, we deduce the following Lagrangian:

$$\max_{s} \log(1-s) + \phi s + \log(Rs) + \lambda R(s - (1-s)).$$

The FOC yields $\frac{1}{1-s} - \phi = \frac{1}{s} + 2R\lambda$. Using program constraints that imply $s = 1/2$, we obtain:

$$\lambda = -\frac{1}{2R}\phi.$$

In absence of externality ($\phi = 0$), we obtain $\lambda = 0$. This confirms that the Euler equation is not a constraint for the planner since private and social invectives for savings are the same. When the planner would like the agent to save less, $\phi < 0$, the planner would like the period 1 marginal utility of the agents to be higher for him to save less. As a consequence, the Lagrange multiplier $\lambda > 0$. In the opposite case, we find $\lambda < 0$ when $\phi < 0$. In general the relationship between the sign of λ and the excess or under saving decision depends on the way the Euler equation is introduced as a constraint.

2 Deriving the Ramsey program

2.1 Rewriting the Ramsey program

The planner's program can be written as:

$$\max_{\left((a_{h,t},c_{h,t},l_{(b,y,e),t})_h,\phi_t,\tau_t\right)_{t\geq 0}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \sum_{h\in\mathcal{H}} S_{h,t}\xi^U_{h,t} \left(U(c_{h,t}, l_{h,t}) 1_{e_h=e} + U(c_{h,t}, \delta_{y_h}) 1_{e_h=u} \right) \right], \tag{2}$$

$$c_{h,t} + a_{h,t} \leq (1+r_t) \sum_{\tilde{h}\in\mathcal{H}} \Pi^a_{\tilde{h}h,t} \frac{S_{\tilde{h},t-1}}{S_{h,t}} a_{\tilde{h},t-1} + \left((1-\tau_t)1_{e_h=e} + \phi_t 1_{e_h=u}\right) l_{h,t} y_h w_t \tag{3}$$

$$\tau_t = \phi_t \frac{S_{u,t}}{S_{e,t}} \tag{4}$$

$$l_{h,t} = \chi^{\varphi}(1-\tau_t)^{\varphi} w_t^{\varphi} y_h^{\varphi} 1_{e_h=e} \tag{5}$$

$$\xi^u_{h,t} U_c\left(c_{h,t}, l_{h,t}\right) - \nu_{h,t} = \beta \mathbb{E}_t \left[(1+r_{t+1}) \sum_{\tilde{h}\in\mathcal{H}} \Pi^u_{h\tilde{h},t+1} \xi^u_{\tilde{h},t+1} U_c\left(c_{\tilde{h},t+1}, l_{\tilde{h},t+1}\right) \right], \tag{6}$$

$$K_t = \sum_h S_{h,t} a_{h,t},\ L_t = \sum_h y_h l_{h,t} S_{h,t},\ w_t = F_L(K_{t-1}, L_t), r_t = F_K(K_{t-1}, L_t). \tag{7}$$

Denoting by $\beta^t S_{h,t}\lambda_{h,t}$ the Lagrange multiplier on the Euler equation, the planner's objective, denoted J for simplicity sake, can be written as:

$$J = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \sum_{h\in\mathcal{H}} S_{h,t}\xi_{h,t}^U \left(U(c_{h,t}, l_{h,t})1_{e_h=e} + U(c_{h,t}, \delta_{y_h})1_{e_h=u}\right)\right] - \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{h\in\mathcal{H}} S_{h,t}\lambda_{h,t}$$
$$\times \left(\xi_{h,t}^u U_c\left(c_{h,t}, l_{h,t}\right) - \beta\mathbb{E}_t\left[(1+r_{t+1})\sum_{\tilde{h}\in\mathcal{H}} \Pi_{h\tilde{h},t+1}^u \xi_{\tilde{h},t+1}^u U_c\left(c_{\tilde{h},t+1}, l_{\tilde{h},t+1}\right)\right]\right),$$

where we have used that $\nu_{h,t}\lambda_{h,t} = 0$. After some rewriting, we obtain:

$$J = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \sum_{h\in\mathcal{H}} S_{h,t}\xi_{h,t}^U U(c_{h,t}, l_{h,t})\right] - \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{h\in\mathcal{H}} S_{h,t}\lambda_{h,t}\xi_{h,t}^u U_c\left(c_{h,t}, l_{h,t}\right)$$
$$+ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^{t+1}(1+r_{t+1}) \sum_{\tilde{h}} \frac{\sum_{h\in\mathcal{H}} S_{h,t}\lambda_{h,t}\Pi_{h\tilde{h},t+1}^u}{S_{\tilde{h},t+1}} S_{\tilde{h},t+1}\xi_{\tilde{h},t+1}^u U_c\left(c_{\tilde{h},t+1}, l_{\tilde{h},t+1}\right)$$

Using the definition: $\Lambda_{\tilde{h},t+1} = \frac{\sum_{h\in\mathcal{H}} S_{h,t}\lambda_{h,t}\Pi_{h\tilde{h},t+1}^u}{S_{\tilde{h},t+1}}$, with $\Lambda_{\tilde{h},0} = 0$, we obtain:

$$J = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{h\in\mathcal{H}} \left(S_{h,t}\xi_{h,t}^U U(c_{h,t}, l_{h,t}) - \left(\lambda_{h,t} - (1+r_t)\Lambda_{h,t}\right)\xi_{h,t}^u U_c\left(c_{h,t}, l_{h,t}\right)\right). \tag{8}$$

2.2 Solving the Ramsey program

The Ramsey program then consists in maximizing J in (8) over $((a_{h,t}, c_{h,t}, l_{h,t})_h, \phi_t, \tau_t)_{t\geq 0}$ subject to constraints (4)–(7). Substituting for ϕ_t, r_t, w_t, we obtain the Lagrangian:

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{h\in\mathcal{H}} S_{h,t}\left(\xi_{h,t}^U \left(U(c_{h,t}, l_{h,t})1_{e_h=e} + U(c_{h,t}, \delta_{y_h})1_{e_h=u}\right)\right. \tag{9}$$
$$\left. - \left(\lambda_{h,t} - (1+F_K(K_{t-1}, L_t))\Lambda_{h,t}\right)\xi_{h,t}^u U_c(c_{h,t}, l_{h,t})\right),$$

with ($\widetilde{\Pi}_{\tilde{h},h,t}^a = \Pi_{\tilde{h},h,t}^a \frac{S_{\tilde{h},t-1}}{S_{h,t}}$):

$$c_{h,t} = -a_{h,t} + (1+F_K(K_{t-1}, L_t))\sum_{\tilde{h}} \widetilde{\Pi}_{\tilde{h},h,t}^a a_{\tilde{h},t-1} \tag{10}$$
$$+ \left((1-\tau_t)1_{e_h=e} + \frac{S_{e,t}}{S_{u,t}}\tau_t 1_{e_h=u}\right)\chi^{\varphi}(1-\tau_t)^{\varphi} y_h^{\varphi+1} F_L(K_{t-1}, L_t)^{1+\varphi}$$

$$l_{h,t} = \chi^{\varphi}(1-\tau_t)^{\varphi} y_h^{\varphi} F_L(K_{t-1}, L_t)^{\varphi} 1_{e_h=e}, \tag{11}$$

$$K_t = \sum_h S_{h,t}a_{h,t},\ L_t = \left(\sum_h S_{h,t}y_h^{\varphi+1}1_{e_h=e}\right)\chi^{\varphi}(1-\tau_t)^{\varphi}F_L(K_{t-1}, L_t)^{\varphi}. \tag{12}$$

The Lagrangian (9), with equations (10)–(12), can be seen as depending only on saving choices $(a_{h,t})$ and replacement rate ϕ_t, In the sections below, we compute the FOC of the Lagrangian with respect to $(a_{h,t})$ (Section 2.3) and replacement rate ϕ_t (Section 2.4).

2.3 FOC with respect to saving choices $a_{h,t}$

We compute the FOC of the Lagrangian (9) with respect to $a_{h,t}$. We will denote:

$$U_{c,h,t} = u'\left(c_{h,t} - \chi^{-1}\frac{l_{h,t}^{1+1/\varphi}}{1+1/\varphi}\right),\ U_{cl,h,t} = -\chi^{-1} l_{h,t}^{\frac{1}{\varphi}} U_{cc,h,t}.$$

$$U_{cc,h,t} = u''\left(c_{h,t} - \chi^{-1}\frac{l_{h,t}^{1+1/\varphi}}{1+1/\varphi}\right),\ U_{l,h,t} = -\chi^{-1} l_{h,t}^{\frac{1}{\varphi}} U_{c,h,t}.$$

We obtain:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial a_{h,t}} &= \sum_{\tilde{h}} S_{\tilde{h},t}\xi^U_{\tilde{h},t} U_{c,\tilde{h},t}\left(\frac{\partial c_{\tilde{h},t}}{\partial a_{h,t}} - \chi^{-1} l_{\tilde{h},t}^{\frac{1}{\varphi}}\frac{\partial l_{\tilde{h},t}}{\partial a_{h,t}}\right) \\
&+ \beta\sum_{\tilde{h}} \mathbb{E}_t\left[S_{\tilde{h},t+1}\xi^U_{\tilde{h},t+1} U_{c,\tilde{h},t+1}\left(\frac{\partial c_{\tilde{h},t+1}}{\partial a_{h,t}} - \chi^{-1} l_{\tilde{h},t+1}^{\frac{1}{\varphi}}\frac{\partial l_{\tilde{h},t+1}}{\partial a_{h,t}}\right)\right] \\
&- \sum_{\tilde{h}} S_{\tilde{h},t}\left(\lambda_{\tilde{h},t} - (1+F_{K,t})\Lambda_{\tilde{h},t}\right)\xi^u_{\tilde{h},t} U_{cc,\tilde{h},t}\left(\frac{\partial c_{\tilde{h},t}}{\partial a_{h,t}} - \chi^{-1} l_{\tilde{h},t}^{\frac{1}{\varphi}}\frac{\partial l_{\tilde{h},t}}{\partial a_{h,t}}\right) \\
&+ \sum_{\tilde{h}} S_{\tilde{h},t}\left(\left(F_{KK,t}\frac{\partial K_{t-1}}{\partial a_{h,t}} + F_{KL,t}\frac{\partial L_t}{\partial a_{h,t}}\right)\right)\Lambda_{\tilde{h},t}\xi^u_{\tilde{h},t} U_{c,\tilde{h},t} \\
&- \beta\sum_{\tilde{h}} \mathbb{E}_t\left[S_{\tilde{h},t+1}\left(\lambda_{\tilde{h},t+1} - (1+F_{K,t+1})\Lambda_{\tilde{h},t+1}\right)\xi^u_{\tilde{h},t+1} U_{cc,\tilde{h},t+1}\left(\frac{\partial c_{\tilde{h},t+1}}{\partial a_{h,t}} - \chi^{-1} l_{\tilde{h},t+1}^{\frac{1}{\varphi}}\frac{\partial l_{\tilde{h},t+1}}{\partial a_{h,t}}\right)\right] \\
&+ \beta\sum_{\tilde{h}} \mathbb{E}_t\left[S_{\tilde{h},t+1}\left(F_{KK,t+1}\frac{\partial K_t}{\partial a_{h,t}} + F_{KL,t+1}\frac{\partial L_{t+1}}{\partial a_{h,t+1}}\right)\Lambda_{\tilde{h},t+1}\xi^u_{\tilde{h},t+1} U_{c,\tilde{h},t+1}\right]
\end{aligned} \tag{13}$$

2.3.1 Partial derivatives

Using equations (10)–(12), we can show for aggregate quantities that:

$$\frac{\partial K_t}{\partial a_{h,t}} = S_{h,t},\ \frac{\partial K_{t-1}}{\partial a_{h,t}} = 0,\ \frac{\partial L_{t+1}}{\partial a_{h,t}} = \frac{\varphi L_{t+1}\frac{F_{KL,t+1}}{F_{L,t+1}}}{1-\varphi L_{t+1}\frac{F_{LL,t+1}}{F_{L,t+1}}} S_{h,t},\ \frac{\partial L_t}{\partial a_{h,t}} = 0.$$

For individual choices, we obtain after some algebra, for labor:

$$\frac{\partial l_{\tilde{h},t}}{\partial a_{h,t}} = 0,\ \frac{\partial l_{\tilde{h},t+1}}{\partial a_{h,t}} = \varphi\frac{\frac{F_{KL,t+1}}{F_{L,t+1}}}{1-\varphi L_{t+1}\frac{F_{LL,t+1}}{F_{L,t+1}}} S_{h,t} l_{\tilde{h},e,t+1} 1_{e_{\tilde{h}}=e},$$

and for consumption:

$$\frac{\partial c_{\tilde{h},t}}{\partial a_{h,t}} = -1_{h=\tilde{h}},$$

$$\frac{\partial c_{\tilde{h},t+1}}{\partial a_{h,t}} = (1+F_{K,t+1})\widetilde{\Pi}^a_{h,\tilde{h},t+1} + \frac{F_{KK,t+1} + \varphi L_{t+1}\frac{F^2_{KL,t+1} - F_{KK,t+1}F_{LL,t+1}}{F_{L,t+1}}}{1-\varphi L_{t+1}\frac{F_{LL,t+1}}{F_{L,t+1}}} S_{h,t}\sum_{h'}\widetilde{\Pi}^a_{h'\tilde{h},t+1}a_{h',t}$$
$$+\left((1-\tau_{t+1})1_{e_{\tilde{h}}=e} + \frac{S_{e,t+1}}{S_{u,t+1}}\tau_{t+1}1_{e_{\tilde{h}}=u}\right) l_{\tilde{h},e,t+1}\tilde{y}\,(1+\varphi)\,\frac{F_{KL,t+1}}{1-\varphi L_{t+1}\frac{F_{LL,t+1}}{F_{L,t+1}}} S_{h,t}.$$

2.3.2 Lagrangian simplification

Using Section 2.3.1, we obtain from the Lagrangian derivative (13) that $\frac{\partial \mathcal{L}}{\partial a_{h,t}} = 0$ iff:

$$\xi^U_{h,t}U_{c,h,t} - (\lambda_{h,t} - (1+F_{K,t})\Lambda_{h,t})\,\xi^u_{h,t}U_{cc,h,t}$$
$$= \beta\sum_{\tilde{h}}\mathbb{E}_t\left[\left(\xi^U_{\tilde{h},t+1}U_{c,\tilde{h},t+1} - \left(\lambda_{\tilde{h},t+1} - (1+F_{K,t+1})\Lambda_{\tilde{h},t+1}\right)\xi^u_{\tilde{h},t+1}U_{cc,\tilde{h},t+1}\right)\right.$$
$$\times\left((1+F_{K,t+1})\widetilde{\Pi}^a_{h,\tilde{h},t+1}\frac{S_{\tilde{h},t+1}}{S_{h,t}} + \frac{F_{KK,t+1} + \varphi L_{t+1}\frac{F^2_{KL,t+1} - F_{KK,t+1}F_{LL,t+1}}{F_{L,t+1}}}{1-\varphi L_{t+1}\frac{F_{LL,t+1}}{F_{L,t+1}}} S_{\tilde{h},t+1}\tilde{a}_{\tilde{h},t+1}\right.$$
$$\left.\left.+\left((1-\tau_{t+1})1_{e_{\tilde{h}}=e} + (1+\varphi)\,\tau_{t+1}\frac{S_{e,t+1}}{S_{u,t+1}}1_{e_{\tilde{h}}=u}\right) S_{\tilde{h},t+1}l_{\tilde{h},e,t+1}\tilde{y}\frac{F_{KL,t+1}}{1-\varphi L_{t+1}\frac{F_{LL,t+1}}{F_{L,t+1}}}\right)\right]$$
$$+\beta\frac{F_{KK,t+1} + \varphi L_{t+1}\frac{F^2_{KL,t+1} - F_{KK,t+1}F_{LL,t+1}}{F_{L,t+1}}}{1-\varphi L_{t+1}\frac{F_{LL,t+1}}{F_{L,t+1}}}\sum_{\tilde{h}}\mathbb{E}_t\left[S_{\tilde{h},t+1}\Lambda_{\tilde{h},t+1}\xi^u_{\tilde{h},t+1}U_{c,\tilde{h},t+1}\right]$$

Using the definition of $\Psi_{h,t} = \xi^U_{h,t}U_{c,h,t} - (\lambda_{h,t} - (1+F_{K,t})\Lambda_{h,t})\,\xi^u_{h,t}U_{cc,h,t}$ and recalling that $\widetilde{\Pi}^a_{\tilde{h},h,t} = \Pi^a_{\tilde{h},h,t}\frac{S_{\tilde{h},t-1}}{S_{h,t}}$, we obtain:

$$\Psi_{h,t} = \beta\sum_{\tilde{h}}\mathbb{E}_t\left[\Psi_{\tilde{h},t+1}\times\left((1+r_{t+1})\Pi^a_{\tilde{h},h,t} + \frac{F_{KK,t+1} + \varphi L_{t+1}\frac{F^2_{KL,t+1} - F_{KK,t+1}F_{LL,t+1}}{F_{L,t+1}}}{1-\varphi L_{t+1}\frac{F_{LL,t+1}}{F_{L,t+1}}} S_{\tilde{h},t+1}\tilde{a}_{\tilde{h},t+1}\right.\right. \quad (14)$$
$$\left.\left.+\left((1-\tau_{t+1})1_{e_{\tilde{h}}=e} + \frac{S_{e,t+1}}{S_{u,t+1}}\tau_{t+1}1_{e_{\tilde{h}}=u}(1+\varphi)\right) S_{\tilde{h},t+1}l_{\tilde{h},e,t+1}\tilde{y}\frac{F_{KL,t+1}}{1-\varphi L_{t+1}\frac{F_{LL,t+1}}{F_{L,t+1}}}\right)\right]$$
$$+\beta\frac{F_{KK,t+1} + \varphi L_{t+1}\frac{F^2_{KL,t+1} - F_{KK,t+1}F_{LL,t+1}}{F_{L,t+1}}}{1-\varphi L_{t+1}\frac{F_{LL,t+1}}{F_{L,t+1}}}\sum_{\tilde{h}}\mathbb{E}_t\left[S_{\tilde{h},t+1}\Lambda_{\tilde{h},t+1}\xi^u_{\tilde{h},t+1}U_{c,\tilde{h},t+1}\right].$$

2.4 FOC with respect to replacement rate ϕ_t

Rather than computing the derivative with respect to ϕ_t, we do so with respect to τ_t (which is equivalent since $\phi_t S_{u,t} = \tau_t S_{e,t}$). We obtain from (9):

$$\frac{\partial \mathcal{L}}{\partial \tau_t} = \sum_{\tilde{h}} S_{\tilde{h},t} \left(\xi^U_{\tilde{h},t} U_{c,\tilde{h},t} - \left(\lambda_{\tilde{h},t} - (1 + F_{K,t}) \Lambda_{\tilde{h},t} \right) \xi^u_{\tilde{h},t} U_{cc,\tilde{h},t} \right) \tag{15}$$

$$\times \left(\frac{\partial c_{\tilde{h},t}}{\partial \tau_t} - (1-\tau_t) \tilde{y} F_{L,t} \frac{\partial l_{\tilde{h},t}}{\partial \tau_t} \right) + \left(\sum_{\tilde{h}} S_{\tilde{h},t} \Lambda_{\tilde{h},t} \xi^u_{\tilde{h},t} U_{c,\tilde{h},t} \right) \left(F_{KK,t} \frac{\partial K_{t-1}}{\partial \tau_t} + F_{KL,t} \frac{\partial L_t}{\partial \tau_t} \right).$$

As for the derivative with respect to $a_{h,t}$, we start with partial derivatives.

2.4.1 Partial derivatives

For aggregate quantities, we obtain quite directly: $\frac{\partial L_t}{\partial \tau_t} = -\frac{\varphi \frac{L_t}{1-\tau_t}}{1 - \varphi L_t \frac{F_{LL,t}}{F_{L,t}}}$, $\frac{\partial K_{t-1}}{\partial \tau_t} = 0$. The computation for individual choices is lengthier and we obtain after some algebra:

$$\frac{\partial l_{\tilde{h},t}}{\partial \tau_t} = -\frac{\frac{\varphi}{1-\tau_t} l_{\tilde{h},t}}{1 - \varphi L_t \frac{F_{LL,t}}{F_{L,t}}} 1_{e_{\tilde{h}}=e},$$

$$\frac{\partial c_{\tilde{h},t}}{\partial \tau_t} = -\frac{\varphi \frac{L_t}{1-\tau_t}}{1 - \varphi L_t \frac{F_{LL,t}}{F_{L,t}}} F_{KL,t} \tilde{a}_{\tilde{h},t} + (1-\tau_t)^{\varphi-1} \frac{S_{e,t}}{S_{u,t}} \chi^{\varphi} \tilde{y}^{\varphi+1} F_{L,t}^{1+\varphi} 1_{e_{\tilde{h}}=u}$$

$$- \left((1-\tau_t) 1_{e_{\tilde{h}}=e} + \frac{S_{e,t}}{S_{u,t}} \tau_t 1_{e_{\tilde{h}}=u} \right) \chi^{\varphi} \tilde{y}^{\varphi+1} (1+\varphi)(1-\tau_t)^{\varphi-1} F_{L,t}^{1+\varphi} \frac{1}{1 - \varphi L_t \frac{F_{LL,t}}{F_{L,t}}}.$$

2.4.2 Back to the Lagrangian expression

Using (15) and Section 2.4.1, we obtain that $\frac{\partial \mathcal{L}}{\partial \tau_t} = 0$ iff:

$$\sum_{\tilde{h}} S_{\tilde{h},t} \Lambda_{\tilde{h},t} \xi^u_{\tilde{h},t} U_{c,\tilde{h},t} = \sum_{\tilde{h}} S_{\tilde{h},t} \Psi_{\tilde{h},t} \tag{16}$$

$$\times \left(-\tilde{a}_{\tilde{h},t} + \frac{l_{\tilde{h},e,t}}{L_t} \tilde{y} \frac{F_{L,t}}{\varphi F_{KL,t}} \left(-(1-\tau_t) 1_{e_{\tilde{h}}=e} + \frac{S_{e,t}}{S_{u,t}} \left(1 - \varphi L_t \frac{F_{LL,t}}{F_{L,t}} - (1+\varphi)\tau_t \right) 1_{e_{\tilde{h}}=u} \right) \right).$$

2.5 Conclusion

We can further simplify the expressions of FOC (14) and (16) using $F(K, L) = K^{\alpha} L^{1-\alpha} - \delta K$.

$$F_L = (1-\alpha) \left(\tfrac{K}{L} \right)^{\alpha} \qquad F_K = \alpha \left(\tfrac{K}{L} \right)^{\alpha-1} - \delta$$

$$F_{KK} = -\alpha (1-\alpha) \tfrac{1}{L} \left(\tfrac{K}{L} \right)^{\alpha-2} \quad F_{LL} = -\alpha (1-\alpha) \tfrac{1}{L} \left(\tfrac{K}{L} \right)^{\alpha} \quad F_{KL} = \alpha (1-\alpha) \tfrac{1}{L} \left(\tfrac{K}{L} \right)^{\alpha-1}$$

This implies: $\frac{F_{KL}^2 - F_{KK}F_{LL}}{F_L} = 0$; $L\frac{F_{LL}}{F_L} = -\alpha$; $\frac{F_L}{F_{KL}L} = \frac{K}{\alpha L}$. We thus deduce after some algebra that (2.3) and (16) become:

$$\frac{\alpha\varphi}{1-\tau_t}\sum_{\tilde{h}} S_{\tilde{h},t}\left(\Lambda_{\tilde{h},t}\xi^u_{\tilde{h},t}U_{c,\tilde{h},t} + \Psi_{\tilde{h},t}\tilde{a}_{\tilde{h},t}\right) = \frac{K_{t-1}}{L_t}\sum_{\tilde{h}} S_{\tilde{h},t}\Psi_{\tilde{h},t}l_{\tilde{h},e,t}\tilde{y} \times \left(\left(\frac{S_{e,t}}{S_{u,t}}1_{e_{\tilde{h}}=u} - 1_{e_{\tilde{h}}=e}\right) + \varphi\frac{S_{e,t}}{S_{u,t}}\frac{\alpha-\tau_t}{1-\tau_t}1_{e_{\tilde{h}}=u}\right), \tag{17}$$

and

$$\begin{aligned}\Psi_{h,t} = {} & \beta\sum_{\tilde{h}}\mathbb{E}_t\left[\Psi_{\tilde{h},t+1}(1+F_{K,t+1})\Pi^a_{h,\tilde{h}}\right] \\ & + \beta\alpha(1-\alpha)\sum_{\tilde{h}}\mathbb{E}_t\left[S_{\tilde{h},t+1}\Psi_{\tilde{h},t+1}l_{\tilde{h},e,t+1}\tilde{y}\frac{1}{L_{t+1}}\left(\frac{K_t}{L_{t+1}}\right)^{\alpha-1}\frac{S_{e,t+1}}{S_{u,t+1}}1_{e_{\tilde{h}}=u}\right] \\ & - \beta\alpha(1-\alpha)\sum_{\tilde{h}}\mathbb{E}_t\left[S_{\tilde{h},t+1}\left(\Lambda_{\tilde{h},t+1}\xi^u_{\tilde{h},t+1}U_{c,\tilde{h},t+1} + \Psi_{\tilde{h},t+1}\tilde{a}_{\tilde{h},t+1}\right)\frac{1}{L_{t+1}}\left(\frac{K_t}{L_{t+1}}\right)^{\alpha-2}\right].\end{aligned} \tag{18}$$

Case $\varphi = 0$. When $\varphi = 0$ (fixed labor supply), it should be pointed out that the FOC (17) simplifies into:

$$\frac{S_{e,t}}{S_{u,t}}\sum_{\tilde{h}} S_{\tilde{h},t}\Psi_{\tilde{h},t}y_{\tilde{h}}1_{e_{\tilde{h}}=u} = \sum_{\tilde{h}} S_{\tilde{h},t}\Psi_{\tilde{h},t}y_{\tilde{h}}1_{e_{\tilde{h}}=e}.$$

Note that depending on the writing of FOC (17), it could also imply for $\varphi = 0$, $\tau_t = 1$, which is obviously not possible. Indeed, in that case, employed agents would earn nothing and be worst off that unemployed agents.

3 Additional quantitative results

We here report the mean and standard deviation of key variables for optimal replacement rate and for the replacement rate fixed at its steady-state value.

4 A decentralization mechanism in the case of explicit partitions

We present here a decentralization mechanism in the case of explicit partitions with an additional pooling mechanism. To simplify the exposition, we present this mechanism in three steps. First, we use the family and island metaphor (see Lucas 1975 and 1990, or Heathcote, Storesletten, and Violante 2016 for a more recent reference) as a direct constraint on the environment. We

	Mean	Standard deviation	
		Optimal ϕ	$\phi = \phi^{SS}$
C	2.9053	0.0497	0.0613
K	47.7902	1.1174	1.2126
Y	4.1000	0.1134	0.1024
L	1.0300	0.0100	0.0162
ϕ	0.7900	0.1621	0
ratio_c	0.8838	0.0925	0.1044

Table 1: Moments of key variables

then derive a recursive decentralization. Finally, we present the risk-sharing arrangement across households, which generates the given truncation. The advantage of this presentation strategy is that the existence of an equilibrium can be proved using standard techniques.

We denote by $N \geq 0$ the length of the truncation for idiosyncratic histories and by S the number of idiosyncratic states (i.e., $S = Card\mathcal{S}$).

4.1 The island metaphor

Island description. There are S^N different islands, corresponding to the S possible idiosyncratic risk realizations. Agents with the same idiosyncratic history for the last N periods are located on the same island. Any island is represented by a vector $s^N = (s^N_{-N+1}, \ldots, s^N_0) \in \mathcal{S}^N$ summarizing the N-period idiosyncratic history of all island inhabitants. At the beginning of each period, agents face a new idiosyncratic shock. Agents with history $\hat{s}^N \in \mathcal{S}^N$ in the previous period are endowed with history s^Nin the current period, and we denote $s^N \succeq \hat{s}^N$ when s^N is a possible continuation of $\hat{s}^N$. The specification $N = 0$ corresponds to the full insurance case (only one island), and thus to the standard representative-agent assumption, which will be used below as a benchmark. Symmetrically, the case $N = \infty$ corresponds to a standard incomplete-market economy with aggregate shocks, in line with Krusell and Smith (1998). To simplify the exposition, we assume that all agents enter the economy with identical initial wealth $(a_{-1,s^N})_{s^N \in \mathcal{S}^N} = a_0$.

The family head. The family head maximizes the welfare of the whole family on all islands, attributing an identical weight to all agents and behaving as a price-taker.[1] The family head can freely transfer resources among agents on the same island, but cannot do so across islands. All agents belonging to the same island are treated identically and will therefore receive the same allocation, as is consistent with welfare maximization. For agents on any island $s^N \in \mathcal{S}^N$, the family head will choose the per capita consumption level $c_{s^N,t}$, the labor supply $l_{s^N,t}$, and the end-of-period savings $a_{s^N,t}$ (remember that capital and public debt are substitutes).

Island sizes. The probability $\Pi^S_{\hat{s}^N s^N,t}$ that an agent with history $\hat{s}^N = (\hat{s}^N_{-N+1}, \ldots, \hat{s}^N_0)$ in period t experiences history $s^N = (s^N_{-N+1}, \ldots, s^N_0)$ in period $t+1$ is the probability of switching from state $\hat{s}^N_0$ at t to state s^N_0 at $t+1$, provided that histories $\hat{s}^N$ and s^N are compatible. Formally, we have $\Pi^S_{\hat{s}^N s^N,t} = 1_{s^N \succeq \hat{s}^N} M_{\hat{s}^N_0, s^N_0}(s_t)$, where $1_{s^N \succeq \hat{s}^N} = 1$ if s^N is a possible continuation of history $\hat{s}^N$ and 0 otherwise. We can thus deduce the law of motion of island sizes $(S_{s^N,t})_{t \geq 0, s^N \in \mathcal{S}^N}$:

$$S_{s^N,t+1} = \sum_{\hat{s}^N \in \mathcal{S}^N} S_{t,\hat{s}^N} \Pi^S_{\hat{s}^N s^N,t}, \tag{19}$$

where the initial size of each island $(S_{-1,s^N})_{s^N \in \mathcal{S}^N}$, with $\sum_{s^N \in \mathcal{S}^N} S_{-1,s^N} = 1$, is given. The law of motion (19) is thus valid from period 0 onwards.

Timing. At the beginning of each period t, agents learn their current idiosyncratic shock and have to move from island $\hat{s}^N$ to island s^N. The family head cannot change the allocation before agents leave the island. As a consequence, agents move by taking their wealth with them, equal to the per capita saving $a_{\hat{s}^N,t-1}$. On island s^N, the wealth of all agents coming from island $\hat{s}^N$ (equal to $S_{\hat{s}^N,t-1} \Pi^S_{\hat{s}^N s^N,t-1} a_{\hat{s}^N,t-1}$) – and for all islands $\hat{s}^N$– is pooled together and then equally divided among the $S_{s^N,t}$ agents of island s^N.[2] Therefore, at the beginning of period t, each agent holds wealth $\tilde{a}_{s^N,t}$ equal to:

$$\tilde{a}_{s^N,t} = \sum_{\hat{s}^N \in \mathcal{S}^N} \frac{S_{\hat{s}^N,t-1}}{S_{s^N,t}} \Pi^S_{\hat{s}^N s^N,t-1} a_{\hat{s}^N,t-1}. \tag{20}$$

[1] As the family head does not internalize the effect of its choice on prices, the allocation is not constrained-efficient, and the distortions identified by Davila et al. (2012) are present in the equilibrium allocation. The planner will reduce them with its instruments in the Ramsey program.

[2] This is the additional pooling mechanism that does not appear in the main text. In practical terms, the aggregation of individual past wealth a^i_{t-1} will rely on different transition probabilities in both cases. In the family setup, the aggregation equation (see (20) below) depends on the transition Π^S, while in the explicit-partition setup, it depends on Π^a (that generally differs from Π^S).

The program of the family head can now be expressed as follows:

$$\max_{\left(a_{s^N,t},c_{s^N,t},l_{s^N,t}\right)_{t\geq 0, s^N\in\mathcal{S}^N}} \mathbb{E}_0 \sum_{t=0}^{\infty}\beta^t\left[\sum_{s^N\in\mathcal{S}^N} S_{s^N,t}\xi^u_{s^N,t}U(c_{s^N,t},l_{s^N,t})\right], \tag{21}$$

$$a_{s^N,t}+c_{s^N,t}=((1-\tau_t)l_{s^N,t}1_{e_0=e}+\phi_t 1_{e_0=u}l_{s^N,e,t})y_0w_t+(1+r_t)\tilde{a}_{s^N,t},\text{ for all } s^N\in\mathcal{S}^N, \tag{22}$$

$$c_{s^N,t},\, l_{s^N,t}\geq 0, a_{s^N,t}\geq -\bar{a},\text{ for all } s^N\in\mathcal{S}^N, \tag{23}$$

$$(S_{-1,s^N})_{s^N\in\mathcal{S}^N}\text{ and } a_0 \text{ are given,} \tag{24}$$

to the law of motion (19) for $(S_{s^N,t})_{t\geq 0}^{s^N\in\mathcal{S}^N}$, and to the definition (20) of $(\tilde{a}_{s^N,t})_{t\geq 0}^{s^N\in\mathcal{S}^N}$.[3] In the budget constraint (22) of island s^N, y_0 and e_0 are the current productivity and employment status respectively, while $l_{s^N,e,t}$ is the labor effort on an island with productivity y_0 but employed (this term is related to the UI scheme).

The family head maximizes aggregate welfare (21) subject to the budget constraints (22) on all islands, to positivity and borrowing constraints (23), and to initial conditions (24). As the objective function is increasing and concave, constraints are linear (i.e., the admissible set is convex), and allocations are bounded (a^{max} guarantees a compact admissible set), the existence of the equilibrium can be proved using standard techniques (see Stokey and Lucas (1989, Chap. 15 and 16)).[4] If $\beta^t\nu_{s^N,t}\mu_t(s^t)$ denotes the Lagrange multiplier of the credit constraint of island s^N, the first-order conditions are:

$$\begin{aligned}&\xi^u_{s^N,t}U_c(c_{s^N,t},l_{s^N,t}1_{e_0=e}+\zeta_{y_0}1_{e_0=u}) \qquad (25)\\ &\quad=\beta\mathbb{E}_t\left[\sum_{\tilde{s}^N\succeq s^N}\Pi^S_{s^N,\tilde{s}^N,t}\xi^u_{s^N,t}U_c(c_{t+1,\tilde{s}^N},l_{\tilde{s}^N,t+1}1_{\tilde{e}_0=e}+\zeta_{y_0}1_{\tilde{e}_0=u})(1+r_{t+1})\right]+\nu_{s^N,t},\end{aligned}$$

$$l_{s^N,t}=\chi^{\varphi}(1-\tau_t)^{\varphi}w_t^{\varphi}y_0^{\varphi}1_{e_0=e}, \tag{26}$$

$$\nu_{s^N,t}(a_{s^N,t}+\bar{a})=0\text{ and }\nu_{s^N,t}\geq 0. \tag{27}$$

To anticipate Section 4.2 below, the first-order conditions (25)–(27) have the same form as those derived in standard incomplete insurance-market models. Although the family head cares about agents moving across islands, the result is similar to that of individuals self-insuring against income risk, due to the law of large numbers.

[3]Note that $\mathbb{E}_t[\cdot]$ in (21) is the expectation operator at date $t\geq 0$ over all future aggregate histories.

[4]Due to the finite heterogeneity representation, we could also prove the existence of a recursive equilibrium. In the interest of conciseness, we do not present this recursive formulation, as it is not necessary for deriving first-order conditions.

Labor market. On any employed island s^N, the market labor supply in efficient units at date t amounts to $y_0 S_{s^N,t} l_{s^N,t}$ (we recall that unemployment agents do not work). Summing across all islands yields the total labor supply:

$$L_t = \sum_{s^N \in \mathcal{S}^N} y_0 S_{s^N,t} l_{s^N,t}. \tag{28}$$

UI scheme budget balance. The budget of the UI scheme has to be balanced at any date and no social debt is allowed. Using individual labor Euler conditions (26), the UI budget constraint becomes: $\phi_t \sum_{(e_0,y_0)\in\mathcal{S}} S_{y_0} y_0^{1+\varphi} 1_{e_0=u} = \tau_t \sum_{(e_0,y_0)\in\mathcal{S}} S_{y_0} y_0^{1+\varphi} 1_{e_0=e}$. Observing that the budget balance actually only depends on the current idiosyncratic state, this can be simplified into $\phi_t \sum_{y_0\in\mathcal{Y}} S_{u,t} S_{y_0} y_0^{1+\varphi} = \tau_t \sum_{y_0\in\mathcal{Y}} S_{e,t} S_{y_0} y_0^{1+\varphi}$, or:

$$\phi_t S_{u,t} = \tau_t S_{e,t}, \tag{29}$$

which is again independent of the partition, as in the main text.

Financial market. Total end-of-period savings of all agents, denoted by A_t at date t are:

$$A_t = \sum_{s^N \in \mathcal{S}^N} S_{s^N,t} a_{s^N,t} = \sum_{s^N \in \mathcal{S}^N} S_{s^N,t+1} \tilde{a}_{s^N,t+1}, \tag{30}$$

where the last equality stems from the pooling equation (20). The clearing of the financial market at date t implies that at any date t, the following equality holds:

$$A_t = B_t + K_t. \tag{31}$$

We can now state our sequential equilibrium definition, which is similar to Aiyagari, Marcet, Sargent, and Seppälä (2002) and Farhi (2010).

Definition 1 (Sequential equilibrium) *A sequential competitive equilibrium is a collection of individual allocations* $(c_{s^N,t}, l_{s^N,t}, \tilde{a}_{s^N,t}, a_{s^N,t})_{t\geq 0}^{s^N\in\mathcal{S}^N}$*, of island population sizes* $(S_{s^N,t})_{t\geq 0}^{s^N\in\mathcal{S}^N}$*, of aggregate quantities* $(L_t, A_t, B_t, K_t)_{t\geq 0}$*, of price processes* $(w_t, r_t)_{t\geq 0}$*, and of a UI policy* $(\tau_t, \phi_t)_{t\geq 0}$*, such that, for an initial distribution of island population and wealth* $(S_{-1,s^N}, a_{-1,s^N})_{s^N\in\mathcal{S}^N}$*, and for initial values of capital stock* $K_{-1} = \sum_{s^N\in\mathcal{S}^N} S_{-1,s^N} a_{-1,s^N}$*, and of the initial aggregate shock* z_{-1}*, we have:*

1. *given prices, individual strategies* $(a_{s^N,t}, c_{s^N,t}, l_{s^N,t})_{t\geq 0}^{s^N\in\mathcal{S}^N}$ *solve the agents' optimization program in equations (21)–(24);*

2. *island sizes and beginning-of-period individual wealth* $(S_{s^N,t}, \tilde{a}_{s^N,t})_{t\geq 0}^{s^N \in \mathcal{S}^N}$ *are consistent with the laws of motion (19) and (20);*

3. *labor and financial markets clear at all dates: for any* $t \geq 0$*, equations (28)–(31) hold;*

4. *the UI budget is balanced and (29) holds at any date;*

5. *factor prices* $(w_t, r_t)_{t\geq 0}$ *are consistent with* $w_t = F_L(Z_t, K_{t-1}, L_t)$ *and* $r_t = F_K(Z_t, K_{t-1}, L_t)$*.*

The equilibrium has a simple structure defined at each date by a finite number of variables and of equations for a given UI policy $(\tau_t, \phi_t)_{t\geq 0}$.

4.2 Decentralization and convergence properties

We now prove that the previous program can be decentralized through fiscal transfers, which are shown to measure the degree of idiosyncratic risk sharing achieved by asset pooling in the island economy. We prove that these transfers – as well as idiosyncratic risk sharing – converge toward zero for large N, under general conditions. We start with given factor prices and without aggregate shocks, and then introduce aggregate shocks below. First, dropping aggregate shocks implies that we have existence proof of a recursive representation (see Huggett 1993). Second, fixing factor prices avoids potential issues relating to equilibrium multiplicity, as shown in Açikgöz (2016) for instance.

The economy is now similar to the previous setup, except for the following differences. First, we consider as given a constant after-tax interest rate r – with $\beta(1 + r) < 1$ – and an after-tax wage w. Second, no family head imposes allocations, and agents are expected-utility maximizers taking UI policy as given. Finally, at each date each agent receives a lump-sum transfer $\Gamma_{N+1}(s^{N+1})$, which is contingent on her individual history s^{N+1} over the previous $N+1$ periods. This is the actual difference compared to a standard incomplete-market framework. Using standard techniques, the agents' program can be written recursively as:[5]

$$V_{N+1}(a, s^{N+1}) = \max_{a',c,l} \xi^u_{s^N} U(c, l) + \beta \mathbb{E}\left[\sum_{(s^{N+1})' \succeq s^{N+1}} \Pi^S_{s^{N+1},(s^{N+1})'} V_{N+1}(a', \left(s^{N+1}\right)')\right], \quad (32)$$

$$a' + c = ((1 - \tau_t) l_{s^N,t} 1_{e_0=e} + \phi_t 1_{e_0=u} l_{s^N,e,t}) y_0 w_t + (1 + r)a + \Gamma_{N+1}(s^{N+1}), \quad (33)$$

$$c,\, l \geq 0,\, a' \geq -\bar{a}, \quad (34)$$

[5]In line with the literature, we denote the savings choice in the current period by a'. The value a is thus the beginning-of-period wealth.

with $l = \zeta_{y_0}$ if $s_0 = (y_0, e_0)$, and where $V_{N+1} : [-\overline{a}, a^{max}] \times \mathcal{S}^{N+1} \to \mathbb{R}$ is the value function, and s_0 the current idiosyncratic shock realization. Compared to the economies studied by Huggett (1993) and Aiyagari (1994), the individual history s^{N+1} is a state variable, as it determines the transfer $\Gamma_{N+1}(s^{N+1})$. The Lagrange multiplier of the credit constraint $a' \geq -\bar{a}$ is denoted ν, and the solution of this program comprises the policy rules g_c^{N+1}, $g_{a'}^{N+1}$, g_l^{N+1}, and g_ν^{N+1} – defined over $[-\overline{a}, a^{max}] \times \mathcal{S}^{N+1}$ – determining consumption, savings, labor supply, and the Lagrange multiplier of the individual budget constraint, respectively. We now propose our characterization result, which states that we can find a particular set of transfers – denoted by $(\Gamma^*_{N+1}(s^{N+1}))_{s^{N+1}\in\mathcal{S}^{N+1}}$ – such that the decentralized economy allocations match those of the family head economy.

Proposition 1 (Finite state space) *There exists a set of balanced transfers, that we denote by $(\Gamma^*_{N+1}(s^{N+1}))_{s^{N+1}\in\mathcal{S}^{N+1}}$, such that any optimal allocation of the family head program (21)–(24) is also a solution to the decentralized program (32)–(34).*

The previous proposition states that the family head program presented in Section 4.1 can be decentralized by the balanced lump-sum transfers $(\Gamma^*_{N+1}(s^{N+1}))_{s^{N+1}\in\mathcal{S}^{N+1}}$ (shortened to Γ^*_{N+1} henceforth). This transfer is formally given in Section 4.3, equation (40). It involves pooling the resources of all agents with the same idiosyncratic history for $N+1$ periods, and redistributing the same amount to agents with the same idiosyncratic history for N periods, such that there are only $(E+1)^N$ possible wealth levels. Thus, the transfers Γ^*_{N+1} mimic the wealth pooling of the island economy, formalized in equation (20), that occurs when agents transfer from one island to another.

As stated in the following proposition, these transfers can be shown to converge to zero for large N.

Proposition 2 (Convergence) *For given factor prices and for a set of transfers equal to Γ^*_{N+1}, if $\kappa \in (0,1)$ and $\overline{N} \geq 1$ exist, such that for all $N \geq \overline{N}$, such that for all $(s_{\bar{N}-1}, \ldots, s_0) \in \mathcal{S}^{\bar{N}}$, and $(\tilde{s}_N, \ldots, \tilde{s}_{\bar{N}}), (\hat{s}_N, \ldots, \hat{s}_{\bar{N}}) \in \mathcal{S}^{N-\bar{N}+1}$ and $a_1, a_2 \in [-\overline{a}, a^{max}]$:*

$$\left|g_{a'}^{N+1}(a_1, (\tilde{s}_N, \ldots, \tilde{s}_{\bar{N}}, s_{\bar{N}-1}, \ldots, s_0)) - g_{a'}^{N+1}(a_2, (\hat{s}_N, \ldots, \hat{s}_{\bar{N}}, s_{\bar{N}-1}, \ldots, s_0))\right| < \kappa\,|a_1 - a_2|, \quad (35)$$

then:

$$\lim_{N\to\infty} \sup_{s^{N+1}\in\mathcal{S}^{N+1}} \left|\Gamma^*_{N+1}(s^{N+1})\right| = 0.$$

The formal proof of the proposition can be found in Section 4.3 below.

Although it may appear complicated, condition (35) has a simple meaning. It states that the marginal propensity to save is always smaller than 1 for all agents, as soon as N is high enough. When this condition is fulfilled, transfers tend toward 0 as the length of idiosyncratic history N increases. If the saving propensity is strictly lower than one, initial differences in wealth vanish and agents experiencing the same history of idiosyncratic shocks end up having the same wealth over time. As a consequence, the wealth pooling generated by the transfers Γ^*_{N+1} concerns wealth levels that tend to be closer to one other, and the transfers Γ^*_{N+1} tend toward 0.

Although condition (35) involves the savings policy function rather than model exogenous parameters, the condition can be shown to hold for a large subclass of HARA utility functions. More precisely, for any HARA utility function of the form $c \mapsto \frac{(c+b)^{1-\sigma}}{1-\sigma}$ (or $c \mapsto \ln(c+b)$), with $c > -b$, $0 < \sigma \neq 1$, we can prove that condition (35) holds with $\kappa = (\beta(1+r))^{\frac{1}{\sigma}} \in (0,1)$.[6] Indeed, following the same steps as Açikgöz (2016), we can show that the savings policy function is a contraction.[7]

Because of the mapping between the island economy and the decentralized one, the absolute size of transfers Γ^*_{N+1} can be thought of as a measure of the insurance provided by the wealth pooling operation during island transfers. As a result, when the length of history becomes infinitely large, the insurance provided by pooling vanishes for long history lengths, as is the case in standard incomplete market economies.

A risk-sharing arrangement. We have achieved decentralization through a set of fiscal transfers Γ^*_{N+1}, providing additional insights into the insurance provided by truncation. This truncation is the outcome of a decentralized equilibrium. Assume that 1) agents have full commitment at period 0; 2) they are ex ante identical (and thus all agree on a risk-reducing mechanism); and 3) they can enter risk-sharing agreements at each period $t \geq N$ with other agents having the same history between period $t-N$ and period t. At each period $t \geq N+1$ agents with the same history for the last N periods insure each other against heterogeneity in idiosyncratic risks prior to period $t-N$. As an equilibrium outcome, agents insure themselves against the heterogeneous realization of the risk at period $t-N-1$ (because risk at periods $t-N-2$ and before has already been insured in period $t-1$). $\Gamma^*_{N+1}(s^{N+1})$ is then the amount

[6] A more general expression for HARA utility functions is $\frac{(\frac{c}{\sigma}+b)^{1-\sigma}}{1-\sigma}$, which includes the CARA utility functions as a limit case when $\sigma \to \infty$. In this case, the proof does not hold.

[7] This is specifically the proof of Açikgöz (2016)'s Proposition 7. The proof relies first on a lemma (Lemma H.1) that can be extended to HARA utility functions. The second part of the proof is based on a result of Jensen (2017) stating that saving policy functions are convex, which also holds for HARA utility functions.

received by an agent with history s^{N+1} from agents with history s^N. As agents are identical in period 0, they all agree to commit to this risk-sharing arrangement, which provides additional but limited insurance. This risk-sharing arrangement mimics the fiscal transfer Γ^*_{N+1}, which in turn mimics the island structure. We do not attempt to provide a microfoundation for this risk-sharing arrangement based on deeper informational frictions (such as the ability to observe agents' past idiosyncratic statuses). In line with the Bewley tradition, we simply use this insurance structure, which provides an intermediate level of insurance between the complete and incomplete insurance-market models, to derive new results about optimal fiscal policies.

4.3 Proof of Proposition 1

Consider an agent endowed with the $N+1$-period history $s^{N+1} = (\hat{s}^N, s) \in \mathcal{S}^{N+1}$, which can also be written as $s^{N+1} = (s_N, s^N)$. In the former notation, s^{N+1} is seen as the history $\tilde{s}^N \in \mathcal{S}^N$ with the successor state $s \in \mathcal{S}$, while in the latter notation, s^{N+1} is seen as the state $e_N \in \mathcal{S}$ followed by history $s^N \in \mathcal{S}^N$. The solutions to the program (32)–(34) are the policy rules $c = g_c^{N+1}(a, s^{N+1})$, $a' = g_{a'}^{N+1}(a, s^{N+1})$, $l = g_l^{N+1}(a, s^{N+1})$ and the multiplier $\nu = g_\nu^{N+1}(a, s^{N+1})$ satisfying:

$$U_c\left(c, l\right) = \beta \mathbb{E}\left[\sum_{s' \in \mathcal{S}} \Pi_{ss'} U_c\left(c', l'\right)\left(1+r\right)\right] + \nu, \tag{36}$$

$$l = \chi^\varphi (1-\tau_t)^\varphi w_t^\varphi y_0^\varphi 1_{e_0=e} + \zeta_y 1_{e_0=u}, \tag{37}$$

$$\nu(a' + \bar{a}) = 0 \text{ and } \nu \geq 0. \tag{38}$$

We use a guess-and-verify strategy. The transfer is constructed such that all agents with the same N-period history have the same after-transfer wealth. The measure of agents with history s^N follows the same law of motion as (19) in the island economy and is equal to S_{s^N}. If agents with the same history $(\hat{s}^N, s), s \in \mathcal{S}$ have the same beginning-of-period wealth $a_{\hat{s}^N}$, the after-transfer wealth, denoted by $\hat{a}_{s^N}$, of agents with history $s^N \succeq \hat{s}^N$ is:

$$\hat{a}'_{s^N} = \sum_{\tilde{s}^N \in \mathcal{S}^N} \frac{S_{\tilde{s}^N}}{S_{s^N}} \Pi_{\tilde{s}^N s^N} a'_{\tilde{s}^N}, \tag{39}$$

such that agents with the same history hold the same wealth. By construction, $\hat{a}_{s^N}$ follows dynamics similar to the "after-pooling" wealth $\tilde{a}_{s^N,t}$ in the island economy of equation (20). The transfer scheme denoted by $\left(\Gamma^*_{N+1}(s^{N+1})\right)_{s^{N+1} \in \mathcal{S}^{N+1}}$ that enables all agents with the same history to have the same wealth is:

$$\Gamma^*_{N+1}(s^{N+1}) = (1+r)\left(\hat{a}_{s^N} - a_{\hat{s}^N}\right), \tag{40}$$

where we use $s^{N+1} = (\hat{s}^N, s) = (s_N, s^N)$. The transfer $\Gamma^*_{N+1}(s^{N+1})$ defined in (40) replaces the beginning-of-period wealth $(1+r)a_{\hat{s}^N}$ with the *average* wealth $(1+r)\hat{a}_{s^N}$. Since there is a continuum with mass $S_{\tilde{s}^N}$ of agents with history $\tilde{s}^N$, in which each individual agent is atomistic, all agents take the transfer Γ^*_{N+1} as given.

Finally, it is easy to check that the transfer scheme is balanced in each period. Using the definition (39) of $\hat{a}_{s^N}$, we obtain for $s^N = (s^N_{N-1}, \ldots, s^N_1, s^N_0) \in \mathcal{S}^N$, $S_{s^N}\hat{a}_{s^N} = \sum_{\hat{s}^N \in \mathcal{S}^N} S_{\hat{s}^N}\Pi_{\hat{s}^N, s^N} a_{\hat{s}^N} = \sum_{\hat{s}\in\mathcal{S}} S_{(\hat{s}, s^N_{N-1}, \ldots, s^N_1)}\Pi_{s^N_1 s^N_0} a_{(\hat{s}, s^N_{N-1}, \ldots, s^N_1)}$. Therefore, we deduce: $\sum_{\tilde{s}\in\mathcal{S}} S_{(\tilde{s}, s^N)}\Gamma^*_{N+1}(\tilde{s}, s^N) = (1+r)\left[\sum_{\tilde{s}\in\mathcal{S}} S_{(\tilde{s}, s^N)}\left(\hat{a}_{s^N} - a_{(\tilde{s}, s^N_{N-1}, \ldots, s^N_1)}\right)\right] = 0$, where the last equality comes from the definition of $\hat{a}_{s^N}$ in equation (39).

4.3.1 Proof of Proposition

The proof comprises three steps. In the remainder, we use the following notation. For $N > k > 0$, $s^k = (s_{k-1}, \ldots, s_0) \in \mathcal{S}^k$, $s^{N,k} = (s_N, \ldots, s_k) \in \mathcal{S}^{N+1-k}$, and $(s^{N,k}, s^k) = (s_N, \ldots, s_k, s_{k-1}, \ldots, s_0)$.

A contraction lemma. We denote by $Conv(A)$ the convex hull of the set $A \subset \mathbb{R}$, and by $\mu_{\mathcal{L}}$ the Lebesgue measure on $\mathbb{R}$.

Lemma 1 (Contraction lemma) *Assume that $A \subset [-\bar{a}, a^{max}]$ and that the conditions of Proposition 2 are fulfilled. Let $B = \left\{g^{N+1}_{a'}(a, (\hat{s}^{N,\bar{N}}, s^{\bar{N}}))|\hat{s}^{N,\bar{N}} \in \mathcal{S}^{N+1-\bar{N}}, a \in A\right\}$ for any $s^{\bar{N}} \in \mathcal{S}^{\bar{N}}$. We then have $\mu_{\mathcal{L}}(Conv(B)) \leq \kappa \times \mu_{\mathcal{L}}(Conv(A))$.*

Proof. Since $B \subset \mathbb{R}$, we have by definition of the convex hull, $Conv(A) = [\min(A), \max(A)]$ and $Conv(B) = [\min(B), \max(B)]$. Let $a' = \max(A)$ and $a = \min(A)$, then $\mu_{\mathcal{L}}(Conv(A)) = a' - a$ and $B \subset [g^{N+1}_{a'}(a, (\hat{s}^{N,\bar{N}}, s^{\bar{N}})), g(a', (\tilde{s}^{N,\bar{N}}, s^{\bar{N}}))]$ for some $\hat{s}^{N,\bar{N}}, \tilde{s}^{N,\bar{N}} \in \mathcal{S}^{N+1-\bar{N}}$. Therefore, we obtain $\mu_{\mathcal{L}}(Conv(B)) \leq g^{N+1}_{a'}(a', (\tilde{s}^{N+1-\bar{N}}, s^{\bar{N}})) - g^{N+1}_{a'}(a, (\hat{s}^{N+1-\bar{N}}, s^{\bar{N}}))$. Applying the Lipschitz property (35) yields $\mu_{\mathcal{L}}(Conv(B)) \leq \kappa \times \mu_{\mathcal{L}}(Conv(A))$. ■

Proof of the convergence of Γ^*_{N+1}. Let $N > 0$. Proposition 1 shows that when the transfer is Γ^*_{N+1}, there are $(E+1)^N$ possible end-of-period asset holdings denoted by $(a'_{\hat{s}^N})_{\hat{s}^N\in\mathcal{S}^N}$. Let $A_{N,-1}$ be the set of all possible end-of-period asset holdings in the previous period. We define:

$$A^{(N)}_N(s^N) = \{a'_{\hat{s}^N} \in A_{N,-1}|s^N \succeq \hat{s}^N\}, \text{ for } s^N \in \mathcal{S}^N, \tag{41}$$

as the set of all possible beginning-of-period and before-transfer asset holdings of agents with current history s^N. In other words, it is the set of all possible previous-period wealth levels of agents with current history s^N. Since the after-transfer wealth level $\hat{a}_{s^N}$ of (39) is an average of before-transfer wealth levels $a'_{\hat{s}^N}$, we have $\hat{a}_{s^N} \in Conv\left(A_N^{(N)}(s^N)\right)$.

We define as $\pi(s^N) = \{\hat{s}^N | s^N \succeq \hat{s}^N\}$ the set of possible predecessors of s^N. We rewrite (41) as: $A_N^{(N)}(s^N) = \{a'_{\hat{s}^N} \in A_{N,-1} | \hat{s}^N \in \pi(s^N)\}$. For any $a'_{\hat{s}^N} \in A_N^{(N)}(s^N)$, there exists $\tilde{s}^N \in \mathcal{S}^N$ such that $\hat{s}^N \succeq \tilde{s}^N$ and $a'_{\hat{s}^N} = g_{a'}^{N+1}(a_{\tilde{s}^N}, (\tilde{s}^N, \hat{s}_0^N)) = g_{a'}^{N+1}(a_{\tilde{s}^N}, (\tilde{s}^N, s_1^N))$ – since $s_1^N = \hat{s}_0^N$. In other words, $a'_{\hat{s}^N}$ is the optimal choice of an agent who, in the previous period, had the N-history $\tilde{s}^N$, which is thus a possible past of s^N. Using the notation $\pi^2 = \pi \circ \pi$, $\tilde{s}^N \in \pi^2(s^N)$, we can define:

$$A_{N-1}^{(N)}(s^N) = \{a'_{\tilde{s}^N} \in A_{N,-2} | \tilde{s}^N \in \pi^2(s^N) \text{ and } g_{a'}^{N+1}(a'_{\tilde{s}^N}, (\tilde{s}^N, s_1^N)) \in A_N^{(N)}(s^N)\},$$

which is the set of all possible end-of-period asset holdings two periods ago of agents with current history s^N. Similarly, we define for any $0 < k < N$:

$$A_{N-k}^{(N)}(s^N) = \{a'_{\tilde{s}^N} \in A_{N,-k-1} | \tilde{s}^N \in \pi^{k+1}(s^N) \text{ and } g_{a'}^{N+1}(a'_{\tilde{s}^N}, (\tilde{s}^N, s_k^N)) \in A_{N-k+1}^{(N)}(s^N)\},$$

which allows us to construct a sequence of sets $(A_{N-k}^{(N)})_{k=0,\ldots,N}$. In other words, $A_{N-k}^{(N)}(s^N)$ is the set of all possible end-of-period asset holdings k periods ago of agents with current history s^N. Iterating backward to construct those sets, we thus go back in time to construct sets of possible wealth levels (instead of histories). In the previous notation, π^{k+1} denotes $\pi \circ \ldots \circ \pi$ ($k+1$ times). Note that we could equivalently define $A_{N-k+1}^{(N)}(s^N)$ as:

$$A_{N-k+1}^{(N)}(s^N) = \{g_{a'}^{N+1}(a'_{\tilde{s}^N}, (\tilde{s}^N, s_k^N)) | \tilde{s}^N \in \pi^{k+1}(s^N) \text{ and } a'_{\tilde{s}^N} \in A_{N-k}^{(N)}(s^N)\}.$$

In other words, $A_{N-k+1}^{(N)}(s^N)$ is the set of successors (with relevant histories) of agents with wealth levels in $A_{N-k}^{(N)}(s^N)$. If $1 \leq k \leq N - \bar{N}$, we deduce, applying Lemma (1), $\mu_{\mathcal{L}}\left(Conv\left(A_{N-k+1}^{(N)}(s^N)\right)\right) \leq \kappa \mu_{\mathcal{L}}\left(Conv(A_{N-k}^{(N)}(s^N))\right)$, and iterating forward:

$$\mu_{\mathcal{L}}\left(Conv(A_N^{(N)}(s^N))\right) \leq \kappa^{N-\bar{N}} \mu_{\mathcal{L}}\left(Conv(A_{\bar{N}}^{(N)}(s^N))\right). \tag{42}$$

Since a^{max} ($-\bar{a}$) is by definition the highest (lowest) wealth level, $A_N \subset [-\bar{a}, a^{max}]$ and for all k, $A_{N-k}^{(N)}(s^N) \subset A_N \subset [-\bar{a}, a^{max}]$. This implies that $\mu_{\mathcal{L}}\left(Conv(A_{\bar{N}}^{(N)}(s^N))\right) \leq a^{max} + \bar{a}$. Second we have shown that $\hat{a}_{s^N}, a_{\hat{s}^N} \in Conv\left(A_N^{(N)}(s^N)\right)$ for any $\hat{s}^N \in \pi(s^N)$, meaning that $|\hat{a}_{s^N} - a_{\hat{s}^N}| \leq \mu_{\mathcal{L}}\left(Conv\left(A_N^{(N)}(s^N)\right)\right)$. As a result, we have from equation (42):$|\hat{a}_{s^N} - a_{\hat{s}^N}| \leq \kappa^{N-\bar{N}}(a^{max} + \bar{a})$, which can be made arbitrarily small ($0 < \kappa < 1$), when N increases. We deduce that $\lim_{N\to\infty} \sup_{s^{N+1}\in\mathcal{S}^{N+1}} |\Gamma^*_{N+1}(s^{N+1})| = 0$.

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